- **Curl Operator:**

$$\vec{\nabla} \times \vec{F} = -\lim_{\Delta V, \Delta S \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \vec{F} \times \overrightarrow{ds}' \quad (1)$$

For an arbitrary unit vector $\hat{a}$ we have

$$\left(\vec{\nabla} \times \vec{F}\right) \cdot \hat{a} = \lim_{\Delta V, \Delta S \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \vec{F} \cdot \left(\hat{a} \times \overrightarrow{ds}'\right) \quad (2)$$

The curl of a vector in an orthogonal coordinate system can be written as:

$$\vec{\nabla} \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \widehat{q_1} & h_2 \widehat{q_2} & h_3 \widehat{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \quad (3)$$

Using the forms of divergence and curl operators in an orthogonal coordinate system we can obtain:

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \quad (4)$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right) = 0 \quad (5)$$

Equations (1) and (2) in turns lead to the following expressions known as the curl theorem:

$$\oint_c \vec{F} \cdot \overrightarrow{dl} = \int_A \left(\vec{\nabla} \times \vec{F}\right) \cdot \overrightarrow{da} \quad (6)$$

$$\oint_S \vec{f} \times \overrightarrow{ds} = -\int_V \vec{\nabla} \times \vec{f} dv \quad (7)$$

Moreover, using the first form for the curl theorem one can easily prove the following integral result:

$$\oint_c \psi \overrightarrow{dl} = -\int_A \vec{\nabla}\psi \times \overrightarrow{da} \quad (8)$$

In a volume V with surrounding surface S, a field vector like $\vec{F}$ can be identified uniquely if we know the divergence and curl of this vector everywhere within V and and its normal component on S.

For electrostatic field $\vec{E}$ everywhere in space one can show that

$$\vec{\nabla} \times \vec{E} = 0 \quad (9)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad (10)$$

Therefore, we can define an electrostatic potential function $\phi(\vec{r})$ such that

$$\vec{E} = -\vec{\nabla}\phi \quad (11)$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' \quad (12)$$

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{13}$$

Last equation known as the Poisson equation for electrostatic problems and one can prove that the solution for this equation in a volume V is unique if we know the ϕ or $\vec{\nabla}\phi \cdot \hat{n}$ on the enclosure surface S.

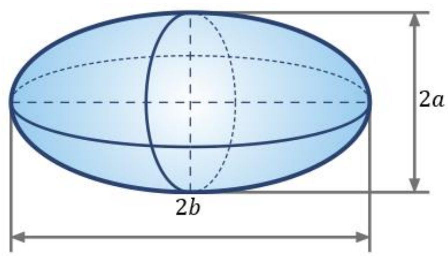
Moreover, equations (12) and (13) interestingly show that

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}') \tag{14}$$

$$\nabla'^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}') \tag{15}$$

HW:

For a uniformly charged spheroidal bunch with total charge q_b, calculate the electrostatic potential function within the volume.



HW:

Prove the mean value theorem, for a charge-free space, the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

HW:

With two different approaches, prove the following equation

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}')$$