Green's Function:

If $\vec{r}$ shows the position vector for the observation point in volume V, then we can write

$$\phi(\vec{r}) = \int_V \phi(\vec{r}')\delta(\vec{r}-\vec{r}')d^3r' \tag{1}$$

Now if we define a two vector variable function like $G(\vec{r}|\vec{r}')$ (known as the Green function) such that

$$\nabla'^2 G(\vec{r}|\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}') \tag{2}$$

We can write

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int_V \phi(\vec{r}')\nabla'^2 G(\vec{r}|\vec{r}')d^3r' \tag{3}$$

Which in turn leads to

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{r}|\vec{r}')\rho(\vec{r}')d^3r' + \frac{1}{4\pi}\oint_S \left[G(\vec{r}|\vec{r}')\vec{\nabla}'\phi(\vec{r}') - \phi(\vec{r}')\vec{\nabla}'G(\vec{r}|\vec{r}')\right]\cdot\overrightarrow{\mathrm{ds}}' \tag{4}$$

The equation (2) for the Green function is a kind of Poisson equation which can have a unique solution provided we know $[\![G(\vec{r}|\vec{r}')]\!]^S$ (Dirichlet) or $[\![\vec{\nabla}'G(\vec{r}|\vec{r}')\cdot\vec{\mathrm{n}}']\!]^S$ (Neumman).

With Dirichlet's boundary condition we have:

$$[\![G(\vec{r}|\vec{r}')]\!]^S = 0 \tag{5}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{r}|\vec{r}')\rho(\vec{r}')d^3r' - \frac{1}{4\pi}\oint_S G(\vec{r}|\vec{r}')\vec{\nabla}'\phi(\vec{r}')\cdot\overrightarrow{\mathrm{ds}}' \tag{6}$$

And for Neumman's boundary condition we have

$$[\![\vec{\nabla}'G(\vec{r}|\vec{r}')\cdot\vec{\mathrm{n}}']\!]^S = -\frac{4\pi}{S} \tag{7}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{r}|\vec{r}')\rho(\vec{r}')d^3r' + \frac{1}{4\pi}\oint_S G(\vec{r}|\vec{r}')\vec{\nabla}'\phi(\vec{r}')\cdot\overrightarrow{\mathrm{ds}}' + \langle\phi\rangle_S \tag{8}$$

Regardless of the governed boundary condition on Green's Function, since we can always write

$$G(\vec{r}|\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r}|\vec{r}') \tag{9}$$

Where $F(\vec{r}|\vec{r}')$ is a solution for the Laplace equation in such away that it helps $\frac{1}{|\vec{r}-\vec{r}'|}$ to satisfy the required boundary equation in problem. Then $\frac{G(\vec{r}|\vec{r}')}{4\pi\varepsilon_0}$ can be consider as the potential function at observation point $\vec{r}$ due to a unit charge located at a point $\vec{r}'$ plus all related image charges with respect to the surface S which can help us to satisfy the required boundary conditions on S.

HW:

The Green function is in general a symmetric function with respect to interchange between $\vec{r}$ and $\vec{r}'$. Prove this symmetry in Dirichlet's boundary condition and find an additional constraint which has to be imposed to the Neumman boundary condition for obtaining this symmetry.

$$G(\vec{r}|\vec{r}') = G(\vec{r}'|\vec{r}) \tag{10}$$