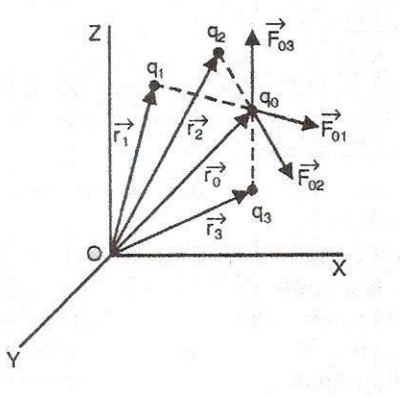
- **Electrostatic Potential Energy:**



The potential energy of a system can be interpreted as the total energy we need to spend for creating the system minus its total kinetic energy at the end. Then if we want to calculate the potential energy of an electrostatic system, we need to calculate the total work that we need to perform to bring N charges from infinity to their known location in the distribution with constant velocity again the electrostatic forces.

In this way, in ith step for moving the ith charge from infinity to $\vec{r}_i$ we have

$$\delta w_i = -\sum_{j=1}^{i-1}\int_{\infty}^{\vec{r}_i}\vec{f}_j(\vec{r}_i{}')\cdot d\vec{r}_i{}' = -q_i\sum_{j=1}^{i-1}\int_{\infty}^{\vec{r}_i}\vec{E}_j(\vec{r}_i{}')\cdot d\vec{r}_i{}'$$

$$= q_i\sum_{j=1}^{i-1}\int_{\infty}^{\vec{r}_i}\vec{\nabla}_i\phi_j(\vec{r}_i{}')\cdot d\vec{r}_i{}' = q_i\sum_{j=1}^{i-1}\phi_j(\vec{r}_i) \qquad (1)$$

Then the total electrostatic potential energy of the system becomes

$$U = \sum_{i=1}^{N}\delta w_i = \sum_{i=1}^{N}\sum_{j=1}^{i-1}q_i\phi_j(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{q_iq_j}{|\vec{r}_i-\vec{r}_j|} \qquad (2)$$

Or in a more simple form as,

$$U = \frac{1}{2}\sum_{i=1}^{N}q_i\phi(\vec{r}_i) = \frac{1}{2}\int\rho(\vec{r})\phi(\vec{r})d^3r \qquad (3)$$

Where

$$\phi(\vec{r}_i) = \sum_{j=1}^{N}\phi_j(\vec{r}_i)\left(1-\delta_j^i\right) \qquad (4)$$

Equation (3) by the use of the Coulomb law can be represented in terms of the electric field as following

$$U = \frac{\varepsilon_0}{2}\int\left|\vec{E}(\vec{r})\right|^2 d^3r \qquad (5)$$

HW:

Using equation (5) in direct integration calculate the potential energy of a system composed of two point charge particles q_1 and q_2 located at $\vec{r}_1$ and $\vec{r}_2$, respectively.

- **Variational Approach in Electrostatic Problems:**

Up to now, we have introduced three approaches for calculation of the electric field and the potential of an electrostatic system, direct integration, Poisson equation and Green function. Here we introduce the fourth approach known as variational approach. First we know from classical mechanic that a system in equilibrium sets in minimum potential energy state. On the other hand, we know from last section that the electrostatic potential energy of system is given by

$$U = \frac{\varepsilon_0}{2}\int \left|\vec{E}(\vec{r})\right|^2 d^3r = \frac{\varepsilon_0}{2}\int \left|\vec{\nabla}\varphi(\vec{r})\right|^2 d^3r \tag{6}$$

Now if we assume a test scaler function as $\psi(\vec{r}|a_i)$ which is a function of position vector $\vec{r}$ and a_is with same boundary condition as the exact potential function $\varphi(\vec{r})$ on S then we can define a functional $U_1(\psi) = U_1(a_i s)$ and write

$$U_1(\psi) = \frac{\varepsilon_0}{2}\int_V \left|\vec{\nabla}\psi\right|^2 d^3r \tag{7}$$

First, we should remind that a function like $f(x)$ has a minimum at point x_0 if the first order of its variations vanishes at this point ($f'(x_0) = 0$) and the second order of its variations become positive number at this point ($f''(x_0) > 0$) since

$$f(x_0 + dx) = f(x_0) + dx f'(x_0) + dx^2 f''(x_0)$$

However, for variations of U in terms of ψ for small variations in ψ within volume V, up to the second order we have

$$\delta U_1(\psi) = \varepsilon_0 \oint_S \delta\psi\vec{\nabla}\psi\cdot\overrightarrow{ds} - \varepsilon_0\int_V \delta\psi\nabla^2\psi d^3r + \frac{\varepsilon_0}{2}\int_V \left|\vec{\nabla}\delta\psi\right|^2 d^3r \tag{8}$$

Now if the boundary condition for this problem is given by $[\![\varphi]\!]^S = cf$, then we should have independent of a_is, $[\![\psi]\!]^S = [\![\varphi]\!]^S = cf$ and so $[\![\delta\psi]\!]^S = 0$. Therefore, the first term in (8) automatically vanishes. On the other hand, the third term in (8), which is our only second order term with respect to $\delta\psi$, is always a positive value. Therefore, minimization of U with respect to a_is will force $[\![\nabla^2\psi]\!]^V = 0$. It means that this minimization process leads to a correct solution for the Laplace equation in volume V with the boundary condition $[\![\varphi]\!]^S = cf$. However, if we want to calculate the solution of the Poisson equation, while in the volume V we have $[\![\nabla^2\varphi]\!]^V = -\frac{\rho}{\varepsilon_0}$, then we can define another version of U i.e. U_2 such that

$$\delta U_2(\psi) = \varepsilon_0 \oint_S \delta\psi\vec{\nabla}\psi\cdot\overrightarrow{ds} - \varepsilon_0\int_V \delta\psi\left[\nabla^2\psi + \frac{\rho}{\varepsilon_0}\right] d^3r + \frac{\varepsilon_0}{2}\int_V \left|\vec{\nabla}\delta\psi\right|^2 d^3r \tag{9}$$

Therefore, clearly minimization of U_2 with respect to a_is leads to a correct solution for the Poisson equation with the boundary condition $[\![\varphi]\!]^S = cf$. Therefore, we should have

$$U_2(\psi) = U_1(\psi) - \varepsilon_0\int_V \psi\frac{\rho}{\varepsilon_0} d^3r \tag{10}$$

Now if we have another type of boundary conditions as $\left[\!\left[\vec{E}\cdot\hat{n}\right]\!\right]^S = cf$, then clearly from (9) we only need to define another functional U_3 such that

$$\delta U_3(\psi) = \varepsilon_0 \oint_S \ \delta\psi\left[\vec{\nabla}\psi + \vec{\mathrm{E}}\right] \cdot \overrightarrow{ds} - \varepsilon_0 \int_V \ \delta\psi\left[\nabla^2\psi + \frac{\rho}{\varepsilon_0}\right] d^3r + \frac{\varepsilon_0}{2}\int_V \ \left|\vec{\nabla}\delta\psi\right|^2 d^3r \qquad (11)$$

And so we obtain following expression as a convenient form in this kind of situation

$$U_3(\psi) = U_2(\psi) + \varepsilon_0 \oint_S \ \psi\vec{\mathrm{E}} \cdot \overrightarrow{ds} \qquad (12)$$

HW:

As a simple application, consider here the two-dimensional problem of a hallow grounded circular cylinder with radius a centered on the z axis with an interior source density $\rho = -\frac{5\varepsilon_0}{a^2}\left(1 - \frac{r}{a}\right) + \frac{10^4\varepsilon_0}{a^2}\left(\frac{r}{a}\right)^5\left(1 - \left(\frac{r}{a}\right)^5\right)$. Then, First: find the exact potential function φ within the volume using numerical calculations with Mathematica, Matlab or etc. Second: find the variational coefficients for a test function $\psi = \alpha\left(\frac{r}{a}\right)^2 + \beta\left(\frac{r}{a}\right)^3 + \gamma\left(\frac{r}{a}\right)^4 - (\alpha + \beta + \gamma)$ which can provide the best fit to the real potential function $\varphi(\vec{r})$ within the volume.