➢ 1-Image Charges:

In real nature many times we involve with electrostatic problems where conductor surfaces paly an essential roles. For instance, in accelerator physics, whole electron gun, beam position monitors, electrostatic coils, electrostatic lenses and etc. In previous section we have considered three different type of methods to encounter with this kind of problems.

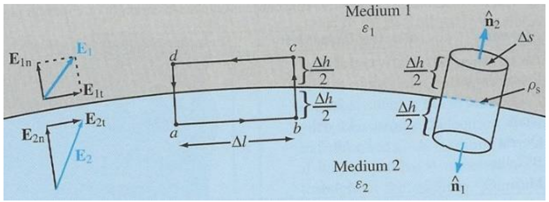
$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' \tag{1}$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{r}|\vec{r}')\rho(\vec{r}')d^3r' + \frac{1}{4\pi}\oint_S \left[G(\vec{r}|\vec{r}')\vec{\nabla}'\varphi(\vec{r}') - \varphi(\vec{r}')\vec{\nabla}'G(\vec{r}|\vec{r}')\right]\cdot\overrightarrow{ds}' \tag{2}$$

$$U_3(\psi) = \frac{\varepsilon_0}{2}\int_V \left|\vec{\nabla}\psi\right|^2 d^3r - \varepsilon_0\int_V \psi\frac{\rho}{\varepsilon_0}d^3r + \varepsilon_0\oint_S \psi\vec{E}\cdot\overrightarrow{ds} \tag{3}$$

In this section we involve with the method of image which is in a very close relation with the method of Green function.

2-Boundary Conditions and the Acting Force on the Surface of a Perfect Conductor:



$$E_{1t} = E_{2t} \rightarrow since\ for\ PC\ \vec{E}_2 = 0 \rightarrow E_{1t} = 0 \tag{4}$$

$$E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_0} \rightarrow since\ for\ PC\ \vec{E}_2 = 0 \rightarrow E_{1n} = \frac{\sigma}{\varepsilon_0} \tag{5}$$

Therfore, on the surfae of a perfect condutor we can write

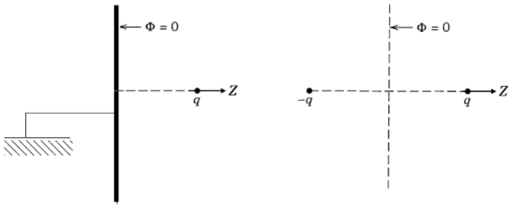
$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \tag{6}$$

Now if we consider an element area ΔA on the surcae of the conductor, then for the force acting on this element we obtain

$$\Delta\vec{F} = \left(\vec{E} - \vec{E}_{self}\right)\Delta q = \left(\vec{E} - \vec{E}_{self}\right)\sigma\Delta A = \left(\frac{\sigma}{\varepsilon_0}\hat{n} - \frac{\sigma}{2\varepsilon_0}\hat{n}\right)\sigma\Delta A = \frac{\sigma^2\Delta A}{2\varepsilon_0}\hat{n} \tag{7}$$

Where $\vec{E}_{self}$ is the self field of the element area which can be calculated easily by the use of Gauss's law and equals to $\frac{\sigma}{2\varepsilon_0}\hat{n}$.

➢ 3-Point Charge in the Presence of a Grounded Perfect Conductor Plane:



In such a situation the electrostatic problems becomes:

$$[\![\nabla^2 \varphi]\!]^{z>0} = -\frac{q}{\varepsilon_0}\delta(\vec{r} - z_0\hat{z}) \tag{8}$$

$$[\![\varphi]\!]^{z=0} = [\![\varphi]\!]^{\{z\to\infty \ or.and \ r\to\infty\}} = 0 \tag{9}$$

Then considering geometry of the problem we can find

$$\varphi = \left(\frac{q}{4\pi\varepsilon_0|\vec{r}-z_0\hat{z}|} - \frac{q}{4\pi\varepsilon_0|\vec{r}+z_0\hat{z}|}\right) \times u(z) \tag{10}$$

And so,

$$E = \left(\frac{q(\vec{r}-z_0\hat{z})}{4\pi\varepsilon_0|\vec{r}-z_0\hat{z}|^3} - \frac{q(\vec{r}+z_0\hat{z})}{4\pi\varepsilon_0|\vec{r}+z_0\hat{z}|^3}\right) \times u(z) \tag{11}$$

With the above equation and using (6) we can calculate the charge distribution on the plane as

$$\sigma = \varepsilon_0 [\![\vec{E}\cdot\hat{z}]\!]^{z=0} = -\frac{q}{2\pi} \times \frac{z_0}{(r^2+{z_0}^2)^{\frac{3}{2}}} = -\frac{q}{2\pi {z_0}^2} \times \frac{1}{\left(1+\left(\frac{r}{z_0}\right)^2\right)^{3/2}} \tag{12}$$

Taking an integration over the last equation we can calculate the total charge distributed on the surface of conductor plane as

$$\mathrm{Q} = \int_0^\infty \sigma\, 2\pi r dr = \frac{-q}{{z_0}^2} \times \int_0^\infty \frac{1}{\left(1+\left(\frac{r}{z_0}\right)^2\right)^{3/2}} r dr = -q \times \int_0^\infty \frac{x dx}{(1+x^2)^{\frac{3}{2}}} = -\frac{q}{2} \times \int_0^\infty \frac{du}{u^{\frac{3}{2}}} = -q \tag{13}$$

It means that the total charge induced on the plane is exactly equal to the artificial image charge.

For this problem one can find following expression for the electrostatic potential energy

$$U = \frac{1}{2}\int \rho\varphi d^3 r - SE = \frac{1}{2} \times \frac{-\mathrm{q}^2}{4\pi\varepsilon_0 \times (2z_0)} \tag{14}$$

Which is exactly equal to the half of the electrostatic potential energy due to a system composed of two point charge $\pm q$ separated by distance of $2z_0$.

For the the total force exerted on the plane by the point charge using (7) we arrive at:

$$\vec{F}_{qp} = \int \frac{\sigma^2 da}{2\varepsilon_0}\hat{z} = \pi\frac{\hat{z}}{\varepsilon_0}\left(\frac{q}{2\pi {z_0}^2}\right)^2 \int_{r=0}^\infty \frac{1}{\left(1+\left(\frac{r}{z_0}\right)^2\right)^3} r dr = \frac{q^2\hat{z}}{4\pi\varepsilon_0 {z_0}^2}\int_{x=0}^\infty \frac{1}{(1+x^2)^3} x dx$$

$$= \frac{q^2\hat{z}}{8\pi\varepsilon_0 {z_0}^2}\int_{u=1}^\infty \frac{1}{u^3} du = \frac{q^2\hat{z}}{4\pi\varepsilon_0 (2z_0)^2} \tag{15}$$

HW: Calculate the potential between two parallel grounded conductors separated in vacuum with a point charge q in between.