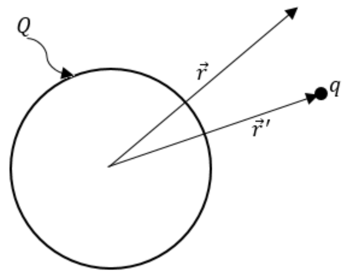
- **Point Charge in the Presence of a Isolated Prfrct Conductor of Total Charge Q:**



For this problem, we can make use of the image method in an iterative procedure as fallowing:

1-In the first stage, we connect the perfect conductor to the ground and bring charge q from infinity to the point $\vec{r}'$. In this situation, a charge $-q\,\frac{a}{r'}$ will be induced on the surface and the total potential of the system becomes

$$\varphi_1(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} - \frac{\mathrm{q}\times\frac{a}{r'}}{4\pi\varepsilon_0\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} \tag{1}$$

2-In the second stage, we should bring another charge from infinity to a specific distance from the sphere (for simplicity in opposite direction with respect to $\vec{r}'$) which can induce charge $Q + \mathrm{q} \times \frac{a}{r'}$ on the sphere. Clearly the charge (Q'') and the distance r'' are

$$-Q'' \times \frac{a}{r''} = Q + \mathrm{q} \times \frac{a}{r'} \tag{2}$$

Then it is enough to have

$$r'' = r' \tag{3}$$

$$Q'' = -\left(Q\,\frac{r'}{a} + \mathrm{q}\right) \tag{4}$$

In this step we have

$$\varphi_2(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} - \frac{\mathrm{q}\times\frac{a}{r'}}{4\pi\varepsilon_0\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases}$$

$$- \frac{\left(Q\frac{r'}{a}+\mathrm{q}\right)}{4\pi\varepsilon_0}\frac{1}{|\vec{r}+\vec{r}'|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} + \frac{\left(Q+\mathrm{q}\times\frac{a}{r'}\right)}{4\pi\varepsilon_0\left|\vec{r}+\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} \tag{5}$$

It is clear that the total charge Q on the sphere has two parts. The first part $-\mathrm{q} \times \frac{a}{r'}$ and the second part $\left(Q + \mathrm{q} \times \frac{a}{r'}\right)$ that are distributed completely separate with distribution functions described in pervious section.

3-In the third stage we disconnect the sphere from earth then on the sphere the total charge Q remains. Now we slowly move charge Q'' to infinity. Since, in this procedure there is no net electric force parallel to the surface then the second part of charge distribution related to total charge $\left(Q + \mathrm{q} \times \frac{a}{r'}\right)$ must redistribute to provide this situation. However, the first part of charge distribution due to the total charge $-\mathrm{q} \times \frac{a}{r'}$ would not be affected. Then the second part of charge distribution slowly and slowly evolves until it would arrive to an uniform distribution. Therefore, we can write

$$\varphi_3(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} - \frac{q\times\frac{a}{r'}}{4\pi\varepsilon_0\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} + \frac{\left(Q+q\times\frac{a}{r'}\right)}{4\pi\varepsilon_0 r} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} \tag{6}$$

And so the electric field becomes

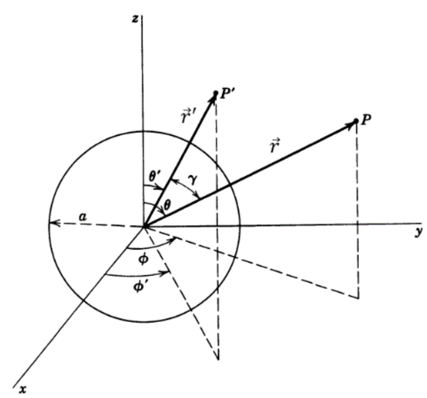
$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} - \frac{q\times\frac{a}{r'}\times\left(\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right)}{4\pi\varepsilon_0\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|^3} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} + \frac{\left(Q+q\times\frac{a}{r'}\right)\times\vec{r}}{4\pi\varepsilon_0 r^3} \times \begin{cases} 1 & r > a \\ 0 & Else \end{cases} \tag{7}$$

Then for the force acting on the point charge we arrive at

$$\begin{aligned}
\vec{F} &= q\left[\frac{\left(Q+q\times\frac{a}{r'}\right)}{4\pi\varepsilon_0 {r'}^2} - \frac{q\times\frac{a}{r'}}{4\pi\varepsilon_0 {r'}^2\left(1-\left(\frac{a}{r'}\right)^2\right)^2}\right]\hat{r}' \\
&= q\left[\frac{\left(Q+\frac{q}{\left(\frac{r'}{a}\right)}\right)}{4\pi\varepsilon_0 a^2\left(\frac{r'}{a}\right)^2} - \frac{q\times\left(\frac{r'}{a}\right)}{4\pi\varepsilon_0 a^2\left(\left(\frac{r'}{a}\right)^2-1\right)^2}\right]\hat{r}' \\
&= \frac{q^2}{4\pi\varepsilon_0 a^2}\left[\frac{\left(\frac{Q}{q}\right)}{\left(\frac{r'}{a}\right)^2} + \frac{1}{\left(\frac{r'}{a}\right)^3} - \frac{\left(\frac{r'}{a}\right)}{\left(\left(\frac{r'}{a}\right)^2-1\right)^2}\right]\hat{r}'
\end{aligned} \tag{8}$$

- **Drichlet Green Function for the Sphere:**

1-Outside:



According to the definition, the Dirichlet Green function for the above system at observation point $\vec{r}$ within volume V (outside space of the sphere) would be the sum potential functions of a unit charge located at point $\vec{r}'$ plus all its image charges outside the volume V with vanishing value on the sphere and infinity. Then clearly, from image approach we can write

$$G(\vec{r}|\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \tag{9}$$

For this function we have

$$[\![\vec{\nabla}' G(\vec{r}|\vec{r}') \cdot \hat{n}']\!]^{r'=a} = -[\![\vec{\nabla}' G(\vec{r}|\vec{r}') \cdot \hat{r}']\!]^{r'=a}$$

$$= -\left[\!\!\left[\frac{\partial}{\partial r'}\left(\frac{1}{\left(r^2+r'^2-2rr'Cos(\gamma)\right)^{1/2}} - \frac{\frac{a}{r'}}{\left(r^2+\left(\frac{a}{r'}\right)^4 r'^2-2r\left(\frac{a}{r'}\right)^2 r'Cos(\gamma)\right)^{1/2}}\right)\right]\!\!\right]^{r'=a}$$

$$= -\left[\!\!\left[\frac{\partial}{\partial r'}\left(\frac{1}{\left(r^2+r'^2-2rr'Cos(\gamma)\right)^{1/2}} - \frac{a}{\left(r'^2r^2+a^4-2rr'a^2Cos(\gamma)\right)^{1/2}}\right)\right]\!\!\right]^{r'=a}$$

$$= \left[\!\!\left[\frac{r'-rCos(\gamma)}{\left(r^2+r'^2-2rr'Cos(\gamma)\right)^{3/2}} - \frac{a\left(r'r^2-ra^2Cos(\gamma)\right)}{\left(r'^2r^2+a^4-2rr'a^2Cos(\gamma)\right)^{3/2}}\right]\!\!\right]^{r'=a}$$

$$= \frac{a^2-r^2}{a\left(r^2+a^2-2raCos(\gamma)\right)^{3/2}} \tag{10}$$

Where

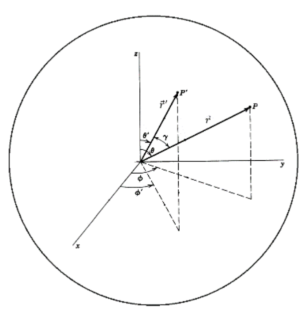
$$Cos(\gamma) = \hat{r}\cdot\hat{r}' = Cos(\theta)Cos(\theta') + Sin(\theta)Sin(\theta')Cos(\phi-\phi') \tag{11}$$

Then we have

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{r>a}\rho(\vec{r}')\left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|}\right)d^3r' + \frac{1}{4\pi}\oint_S \left[G(\vec{r}|\vec{r}')\vec{\nabla}'\varphi(\vec{r}') - \varphi(\vec{r}')\vec{\nabla}'G(\vec{r}|\vec{r}')\right]\cdot\overrightarrow{ds}'$$

$$= \frac{1}{4\pi\epsilon_0}\int_{r>a}\rho(\vec{r}')\left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|}\right)d^3r' + \frac{a}{4\pi}\oint_{Sphere}\varphi(a\hat{r}')\frac{r^2-a^2}{\left(r^2+a^2-2raCos(\gamma)\right)^{3/2}}d\Omega' \tag{12}$$

2-Inside:



In this situation for the Grean function we should have similarly

$$G(\vec{r}|\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|} \tag{13}$$

For this function using (13) we obtain

$$\left[\!\left[\vec{\nabla}' G(\vec{r}|\vec{r}')\cdot\hat{n}'\right]\!\right]^{r'=a} = \left[\!\left[\vec{\nabla}' G(\vec{r}|\vec{r}')\cdot\hat{r}'\right]\!\right]^{r'=a}$$

$$= -\frac{a^2-r^2}{a\left(r^2+a^2-2raCos(\gamma)\right)^{3/2}} \tag{14}$$

Then

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{r>a}\rho(\vec{r}')\left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|}\right)d^3r' + \frac{1}{4\pi}\oint_S \left[G(\vec{r}|\vec{r}')\vec{\nabla}'\varphi(\vec{r}') - \varphi(\vec{r}')\vec{\nabla}'G(\vec{r}|\vec{r}')\right]\cdot\overrightarrow{ds}'$$

$$= \frac{1}{4\pi\epsilon_0}\int_{r>a}\rho(\vec{r}')\left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{\frac{a}{r'}}{\left|\vec{r}-\left(\frac{a}{r'}\right)^2\vec{r}'\right|}\right)d^3r' + \frac{a}{4\pi}\oint_{Sphere}\varphi(a\hat{r}')\frac{r^2-a^2}{\left(r^2+a^2-2raCos(\gamma)\right)^{3/2}}d\Omega' \tag{15}$$

HW: Calculate the Potential function Outside of a Conducting Sphere with Hemispheres at Different Potentials: