Orthogonal Functions and Expansion:

The representation of solution of potential problems (or any mathematical physics problem) by expansions in orthogonal functions forms A POWERFUL TECHNIQUE THAT CAN BE USED IN LARGE CLASS OF PROBLEMS. The particular orthogonal set chosen depends on the symmetry or near symmetry involved.

Let's consider a set of orthogonal functions $U_n(x)$ such that in an interval $[a, b]$ we can expand any well-behaved function $f(x)$ as following.

$$f(x) = \sum_{n=1}^{N} a_n U_n(x) \tag{1}$$

With following property

$$\int_a^b U_n(x) U_m{}^*(x)\, dx = \delta_m^n \tag{2}$$

Using (10-2) for (10-1) leads to

$$\int_a^b f(x) U_n(x)^*\, dx = \int_a^b \sum_{m=1}^{N} a_m U_m(x)\, U_n(x)^*\, dx = a_n \tag{3}$$

Moreover, one can show that

$$\sum_{n=1}^{N} U_n(x') U_n(x)^* = \delta\left(x' - x\right) \tag{4}$$

Which is known as the completeness relation.

It is clear that if we want to have a complete set of orthogonal functions which can expand any arbitrary well-behaved function within the defined interval, it must have infinite number of orthogonal functions.

On the other hand, a complete set of orthogonal functions for a finite interval is discreet otherwise it would be continues with following properties

$$f(x) = \int_{-\infty}^{+\infty} a(k) U(x|k) dk \tag{5}$$

$$a(k) = \int_{-\infty}^{+\infty} f(x) U(x|k)^* dx \tag{6}$$

And completeness relations as

$$\delta(k - k') = \int_{-\infty}^{+\infty} U(x|k')^* U(x|k) dx \tag{7}$$

$$\delta(x - x') = \int_{-\infty}^{+\infty} U(x'|k)^* U(x|k) dk \tag{8}$$

There are infinite number of complete sets of orthogonal functions. Some important ones with their defined interval are as following

1- $\sqrt{\frac{2}{L}} Sin\left(\frac{n\pi x}{L}\right)$ with $n = 1,2,3, \cdots$in the interval $[0, L]$ for all functions with vanishing values at the boundaries.

$$\int_0^L \sqrt{\frac{2}{L}} Sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} Sin\left(\frac{n'\pi x}{L}\right) dx = \delta_{n'}^n \tag{9}$$

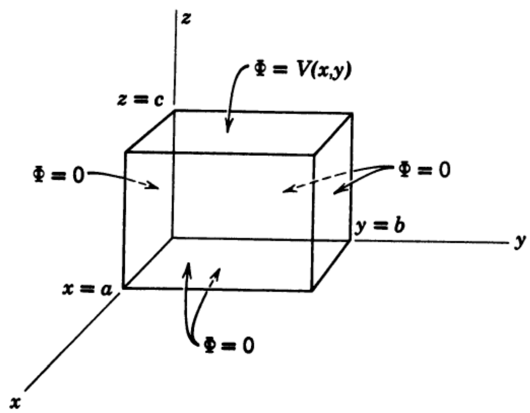
2- $\sqrt{\frac{1}{L}}e^{\left(\frac{2in\pi x}{L}\right)}$ with $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ in the interval $[0, L]$ for all functions with equal values at the boundaries.

$$\int_0^L \sqrt{\frac{1}{L}}e^{\left(\frac{2in\pi x}{L}\right)}\sqrt{\frac{1}{L}}e^{\left(\frac{2in'\pi x}{L}\right)}dx = \delta_{n'}^{n} \tag{10}$$

3- $\sqrt{\frac{1}{2\pi}}e^{ikx}$ in the interval $[-\infty, +\infty]$ for all square integral functions in this interval.

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{\sqrt{2\pi}}\frac{e^{-ik'x}}{\sqrt{2\pi}}dx = \delta(k - k') \tag{11}$$

- **Laplas Equation in Rectangular Coordinate:**



Since the set of $\sqrt{\frac{2}{L}}Sin\left(\frac{n\pi x}{L}\right)$ with $n = 1,2,3, \cdots$ in the interval $[0, L]$ for all functions with vanishing values at the boundaries, provide a complete set for expansion then we can write

$$\varphi(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}(z)\sqrt{\frac{2}{a}}Sin\left(\frac{n\pi x}{a}\right)\sqrt{\frac{2}{b}}Sin\left(\frac{m\pi y}{b}\right) \tag{12}$$

On the other hand, according to the Laplace equation we should have

$$\nabla^2\varphi(x, y, z) = 0 \rightarrow \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{\frac{2}{a}}Sin\left(\frac{n\pi x}{a}\right)\sqrt{\frac{2}{b}}Sin\left(\frac{m\pi y}{b}\right)\left[\frac{d^2}{dz^2} - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2\right]A_{n,m}(z) = 0 \tag{13}$$

Results in

$$\left[\frac{d^2}{dz^2} - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2\right]A_{n,m}(z) = 0 \tag{14}$$

With $A_{n,m}(0) = 0$, and so

$$A_{n,m}(z) = A_{n,m}Sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\,z\right) \tag{15}$$

The final coefficients $A_{n,m}$ can be obtained through boundary condition on $z = c$ i.e.

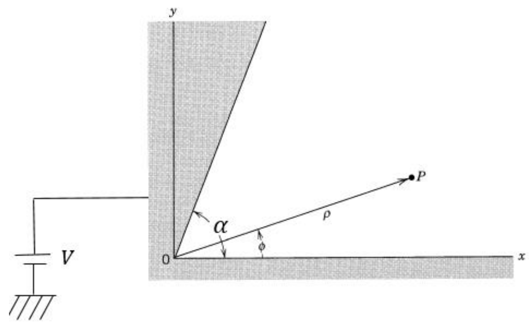
$$V(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m} Sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c\right) \qquad (16)$$

From the last equation, finally we obtaine

$$A_{n,m} = \frac{\int_0^a \int_0^b \sqrt{\frac{2}{a}} Sin\left(\frac{n\pi x}{a}\right) \sqrt{\frac{2}{b}} Sin\left(\frac{m\pi y}{b}\right) V(x,y)\,dx\,dy}{Sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c\right)} \qquad (17)$$

- **Fields and Charge Densities in Two-Dimensional Corners and Along Edges:**

In many practical situations conducting surfaces come together in a way that can be approximated, on the small scale at least, as the intersection of two planes. It is useful to have understanding of how the potential fields, and the surface-charge densities behave in the neighborhood of such sharp corners or edges. Tobe able to look at them closely enough to have the behavior of the fields determined in functional form solely by the properties of the corner and not by the details of overall configuration, we assume that the corners are infinitely sharp.



For this problem, care must be taken that:

First, clearly, here since in infinity the potential doesn't converge to zero therefore, it is an open boundary condition electrostatic problem which cannot be determined uniquely.

Second, the configuration has a symmetry with respect to z and so we can write $\varphi = \varphi(r, \emptyset)$.

Third, the potential function can be composed of two separate parts of V plus another function that vanishes on the boundaries.

Therefore, since the set of $\sqrt{\frac{2}{\alpha}} Sin\left(\frac{n\pi\emptyset}{\alpha}\right)$ with $n = 1,2,3, \cdots$in the interval $[0, \beta]$ for all functions with vanishing values at the boundaries, provide a complete set for expansion then we can write

$$\varphi(r, \emptyset) = V + \sum_{n=1}^{\infty} A_n(r) \sqrt{\frac{2}{\alpha}} Sin\left(\frac{n\pi\emptyset}{\alpha}\right) \qquad (18)$$

On the other hand, using Laplace equation we obtain

$$\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{1}{r^2}\left(\frac{n\pi}{\alpha}\right)^2\right] A_n(r) = \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\left(\frac{n\pi}{\alpha}\right)^2\right] A_n(r) = 0 \qquad (19)$$

Therfore,

$$A_n(r) \propto r^m \tag{20}$$

And so

$$\left[m(m-1) + m - \left(\frac{n\pi}{\alpha}\right)^2\right] = 0 \rightarrow m = \pm\frac{n\pi}{\alpha} \tag{21}$$

On the other hand, in the origin we shoeld have finite value for the potential then

$$A_n(r) = A_n r^{\frac{n\pi}{\alpha}} \tag{22}$$

Then for $\varphi(r, \emptyset)$ we arrive at

$$\varphi(r, \emptyset) = V + \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\alpha}} \sqrt{\frac{2}{\alpha}} Sin\left(\frac{n\pi\emptyset}{\alpha}\right) \tag{23}$$

However, what is important for us is the field behavior near the corner. To this end with taking gradient from (23) we obtain

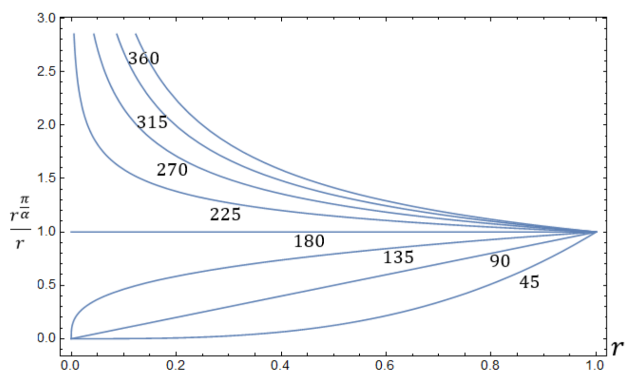
$$\vec{E}(r, \emptyset) = -\sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{\alpha}} \frac{n\pi}{\alpha} \frac{r^{\frac{n\pi}{\alpha}}}{r} \left\{\hat{r} Sin\left(\frac{n\pi\emptyset}{\alpha}\right) + \hat{\emptyset} Cos\left(\frac{n\pi\emptyset}{\alpha}\right)\right\} \tag{24}$$

And so

$$\sigma(r, 0) = \epsilon_0 \vec{E}(r, 0) \cdot \hat{\emptyset}(0) = -\epsilon_0 \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{\alpha}} \frac{n\pi}{\alpha} \frac{r^{\frac{n\pi}{\alpha}}}{r} \tag{25}$$

$$\sigma(r, \alpha) = -\epsilon_0 \vec{E}(r, \alpha) \cdot \hat{\emptyset}(\alpha) = \epsilon_0 \sum_{n=1}^{\infty} (-1)^n A_n \sqrt{\frac{2}{\alpha}} \frac{n\pi}{\alpha} \frac{r^{\frac{n\pi}{\alpha}}}{r} \tag{26}$$

Therefore, on the both sides for very small r we have $\sigma \propto \frac{r^{\frac{\pi}{\alpha}}}{r}$. In the following figure we presented the diagrams of $\eta = \frac{r^{\frac{\pi}{\alpha}}}{r}$ with respect to r for different α with the step of $\frac{\pi}{4}$.



With increasing α from zero η will increase till arrive to $\alpha = 180$ which presents a plane with uniform distribution independent of r. For values of α larger than π where we obtain sharper and sharper corner η will increase further up to its maximum value at $\alpha = 360$.

Then very close to the edge, the corners with smaller α show larger charge distribution and so more charges will concentrate on the sharper edges.

Q: how much is the limit of the electric field and line charge distribution on the edge?

A: Since $\left|\vec{E}(r \to 0, \emptyset)\right| \propto \eta = \frac{r^{\frac{\pi}{\alpha}}}{r}$, then the electric field tends to zero for $\alpha < \pi$ and tends to infinity for $\alpha > \pi$ sharp edges.

But for the line charge distribution have

$$\lambda = \int_{r=0}^{\delta r} \sigma dr \propto \int_{r=0}^{\delta r} \frac{r^{\frac{\pi}{\alpha}}}{r} dr = \frac{\alpha(\delta r)^{\frac{\pi}{\alpha}}}{\pi} \tag{27}$$

Which interestingly presents finite value for the total charge distributed on the corner.

HWs: Problems 2.7, 2.13, 2.15 in Jackson 3rd edition.