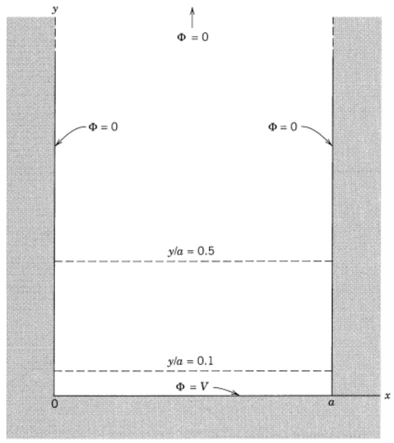
- **A Two-Dimensional Potential Problem; Summation of Fourier Series:**



To solve this problem since the set of $\sqrt{\frac{2}{a}} Sin\left(\frac{n\pi x}{a}\right)$ with $n = 1,2,3,\cdots$in the interval $[0, a]$ for all functions with vanishing values at the boundaries, provide a complete set for expansion then we can write

$$\varphi(x,y) = \sum_{n=1}^{\infty} A_n(y)\sqrt{\frac{2}{a}} Sin\left(\frac{n\pi x}{a}\right) \tag{1}$$

On the other hand, according to the Laplace equation we should have

$$\left[\frac{d^2}{dy^2} - \left(\frac{n\pi}{a}\right)^2\right] A_n(y) = 0 \tag{2}$$

with $\varphi(x, y \to \infty) = 0$, we obtain

$$A_n(y) = A_n e^{\frac{-n\pi}{a} y} \tag{3}$$

On the other hand, we since $\varphi(x, 0) = V$, then

$$V = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} Sin\left(\frac{n\pi x}{a}\right) \tag{4}$$

leads to

$$V\sqrt{\frac{2}{a}} \frac{-a}{n\pi} \left[\!\left[Cos\left(\frac{n\pi x}{a}\right) \right]\!\right]_0^a = A_n \to A_n = \frac{V\sqrt{2a}}{n\pi}[1 - Cos(n\pi)] \tag{5}$$

And so

$$\varphi(x,y) = \sum_{n=1}^{\infty} \frac{2V}{n\pi}[1 - Cos(n\pi)] e^{\frac{-n\pi}{a} y} Sin\left(\frac{n\pi x}{a}\right) = \sum_{n=0}^{\infty} \frac{4V}{(2n+1)\pi} e^{-\frac{(2n+1)\pi}{a} y} Sin\left(\frac{(2n+1)\pi x}{a}\right) \tag{6}$$

Clearly the above equation would converge very fast for $y > a/\pi \cong a/3$ since all higher order terms are at least $e^{-2} = 0.13$ times smaller.

However, to have more simplified expersion for the potential function, from the last equation we can write

$$\varphi(x,y) = \frac{4V}{\pi} \mathcal{I}m\left\{\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \left(e^{\frac{i\pi}{a}(x+iy)}\right)^{(2n+1)}\right\} \tag{7}$$

If we define a complex variable $z = e^{\frac{i\pi}{a}(x+iy)}$ then we can write

$$\varphi = \varphi(z) = \frac{4V}{\pi} im\left\{\sum_{n=0}^{\infty} \frac{1}{(2n+1)} z^{(2n+1)}\right\}$$

$$= \frac{4V}{\pi} im\{\int_0^z \sum_{n=0}^{\infty} {z'}^{2n} dz'\}$$

$$= \frac{4V}{\pi} im\left\{\int_0^z \sum_{n=0}^{\infty} \left({z'}^2\right)^n dz'\right\}$$

$$= \frac{4V}{\pi} im\left\{\int_0^z \frac{dz'}{1-{z'}^2}\right\}$$

$$= \frac{4V}{\pi} im\{Tanh^{-1}(z)\} \quad (8)$$

Moreover, if we define a complex varable $w = Tanh^{-1}(z)$ then

$$z = Tanh(w) \quad (9)$$

Therfore,

$$z = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{e^{2w}-1}{e^{2w}+1} \quad (10)$$

And so

$$e^w = \sqrt{\frac{1+z}{1-z}} \quad (11)$$

Which leads to,

$$w = \frac{1}{2}\ln\frac{1+z}{1-z} = \frac{1}{2}\ln\frac{1-|z|^2+2i\,im(z)}{|1-z|^2}$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{(1-|z|^2)^2+\left(2im(z)\right)^2}}{|1-z|^2} e^{iTan^{-1}\frac{2im(z)}{1-|z|^2}}\right)$$

$$= \frac{1}{2}\left[\ln\left(\frac{\sqrt{(1-|z|^2)^2+\left(2im(z)\right)^2}}{|1-z|^2}\right) + iTan^{-1}\frac{2im(z)}{1-|z|^2}\right] \quad (12)$$

using (12) for (9) yeals

$$im\{Tanh^{-1}(z)\} = im\{w\} = \frac{1}{2}Tan^{-1}\frac{2im(z)}{1-|z|^2} \quad (13)$$

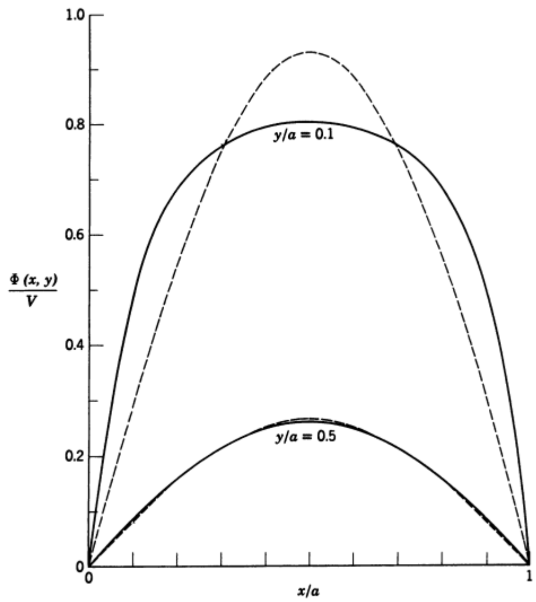
On the other hand, since $z = e^{\frac{i\pi}{a}(x+iy)}$, we obtain

$$im\{Tanh^{-1}(z)\} = \frac{1}{2}Tan^{-1}\left(\frac{2im\left(e^{\frac{i\pi}{a}(x+iy)}\right)}{1-\left|e^{\frac{i\pi}{a}(x+iy)}\right|^2}\right) = \frac{1}{2}Tan^{-1}\left(\frac{2e^{-\frac{\pi y}{a}}Sin\left(\frac{\pi x}{a}\right)}{1-e^{-2\frac{\pi y}{a}}}\right)$$

$$= \frac{1}{2}Tan^{-1}\left(\frac{Sin\left(\frac{\pi x}{a}\right)}{Sinh\left(\frac{\pi y}{a}\right)}\right) \quad (14)$$

Then by substituting (14) into (8) we obtain

$$\varphi(x,y) = \frac{2V}{\pi} Tan^{-1}\left(\frac{Sin\left(\frac{\pi x}{a}\right)}{Sinh\left(\frac{\pi y}{a}\right)}\right) \tag{15}$$



In the above expression, the dashed line represents the first term in the series expansion (7) but the solid line shows the exact solution (15).