Systematic Analysis of Flow Distributions

Heavy-Ion Collisions

arXiv:2006.16019

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IPM

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Standard model of heavy ion collision

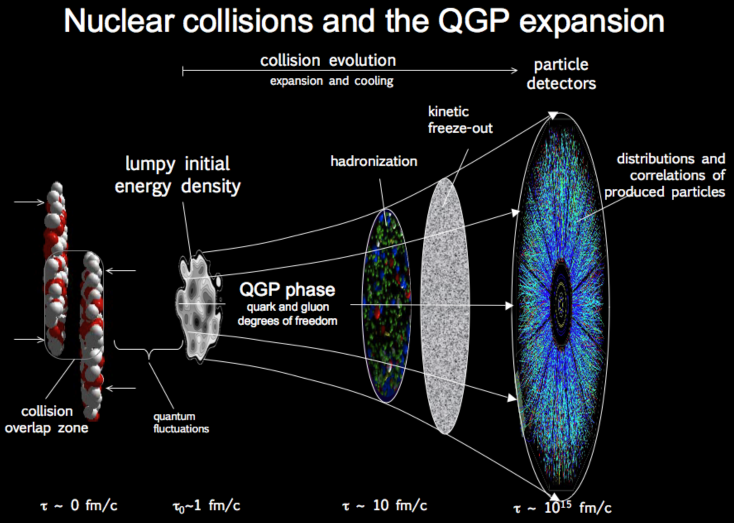
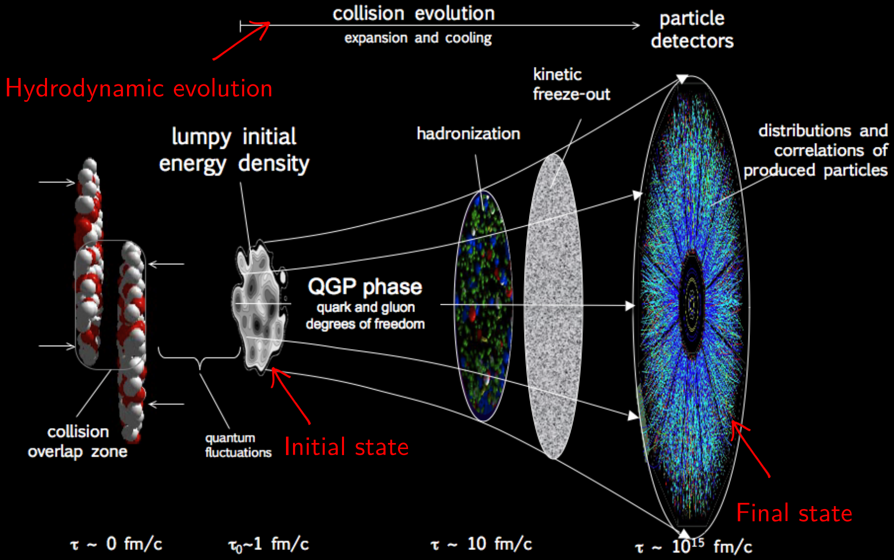


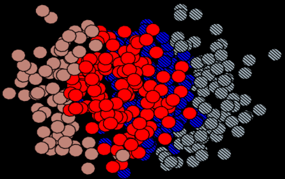
Figure from: Sorensen, arXiv: 0905.0174

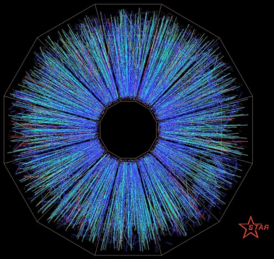
$1fm = 10^{-15}m$

Nuclear collisions and the QGP expansion



Initial Geometry and Final Distribution





- Initial anisotropy:

$$\mathcal{E}_n = \varepsilon_n e^{in\Phi_n} \equiv -\frac{\{r^n e^{in\varphi}\}}{\{r^n\}}$$

$$\{\cdots\} = \int \cdots \rho(r,\phi) d^2 r_\perp$$

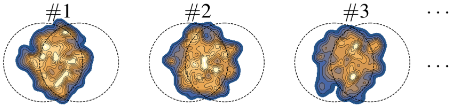
- Anisotropic flow:

$$P(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} V_n e^{-in\phi}$$

$$V_n = v_n e^{in\psi_n} \implies v_n = |V_n|$$

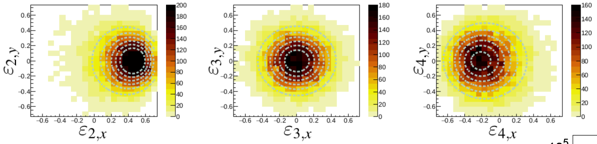
Event-By-Event Fluctuations

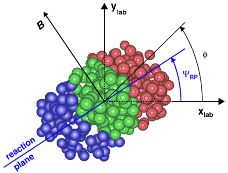
50-55% centralities

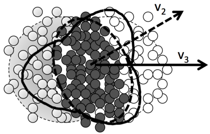


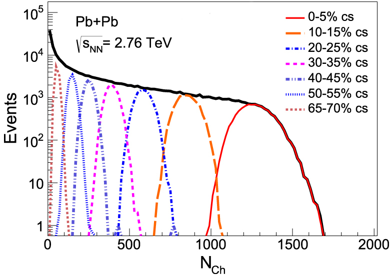
$dN_{\text{events}}/d\varepsilon_{n,x}d\varepsilon_{n,y}$

$\mathcal{E}_n \quad \& \quad V_n$









Anisotropic flow

[Borghini, Dinh, Ollitrault, 2000]

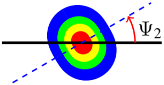
- Fourier analysis of azimuthal distribution of emitted particles

$$\frac{2\pi}{N}\frac{dN}{d\phi} = 1 + 2\sum_{n=1}^{\infty} v_n(p_T, y)\cos[n(\phi - \psi_n)],$$

$$v_n e^{in\psi_n} \propto \langle e^{in\phi}\rangle \propto \int d\phi \frac{dN}{d\phi} e^{in\phi}$$

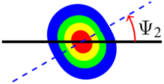
Flow Analysis

- The Fourier coefficients can be found by fitting the expansion to experimental data, but ...



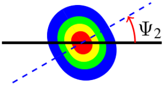
Flow Analysis

- The Fourier coefficients can be found by fitting the expansion to experimental data, but ...
- These enforce us to use ...



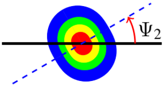
Flow Analysis

- The Fourier coefficients can be found by fitting the expansion to experimental data, but ...
- These enforce us to use ...
- Finding the flow harmonics is not straightforward, because ...



Flow Analysis

- The Fourier coefficients can be found by fitting the expansion to experimental data, but ...
- These enforce us to use ...
- Finding the flow harmonics is not straightforward, because ...
- There are several methods to find flow harmonics experimentally ...



Flow Harmonics, n-Particle Correlation Function

[Borghini, Dinh, Ollitrault, 2000]

- Define 2-particle correlation function

$$c_n\{2\} = \langle e^{in(\phi_1-\phi_2)}\rangle_{\text{single then many events}}$$

- Instead of using $e^{in\phi}$ we use $c_n\{2\}$,

$$\int d\phi_1 d\phi_2 (\frac{dN}{d\phi_1 d\phi_2}) e^{in(\phi_1-\phi_2)} \approx (\int d\phi_1 (\frac{dN}{d\phi_1}) e^{in\phi_1})(\int d\phi_2 (\frac{dN}{d\phi_2}) e^{in\phi_2}) \propto v_n^2$$

- As a result

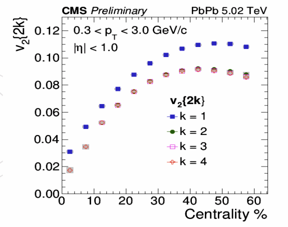
$$v_n^2\{2\} = c_n\{2\}$$

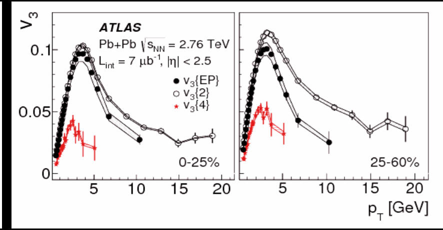
[Borghini, Dinh, Ollitrault, 2000], [Borghini, Dinh, Ollitrault, 2001]

- Generalizing the 2-Particle to n-Particle Correlation Function

$c_n\{2\} = \langle e^{in(\phi_1-\phi_2)}\rangle$, $c_n\{4\} = \langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}\rangle - 2\langle e^{in(\phi_1-\phi_2)}\rangle^2$, ...

- It is shown that
  $v_n^2\{2\} = c_n\{2\}$, $v_n^4\{4\} = -c_n\{4\}$, $v_n^6\{6\} = c_n\{6\}/4$, ...
- Splitting in $v_n\{2k\}$

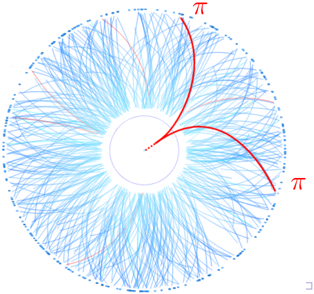




Why flow distribution?

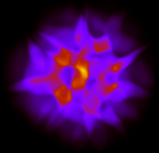
The flow fluctuations contain the information of the collision geometry, quantum fluctuations at the initial state, and effects of different evolution stages in the heavy-ion process.

- Low multiplicity in a single event
- Randomness of the reaction plane angle
- Non-flow effects

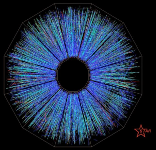


Evolution Response

Hydrodynamic equations



$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^\mu = 0$$



- Initial anisotropy: $\mathcal{E}_n = \varepsilon_n e^{in\Phi_n}$

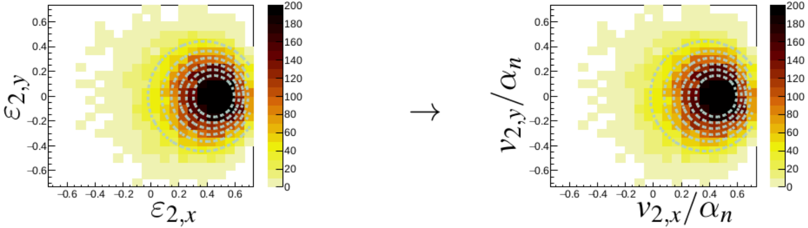
$$V_n = V(\mathcal{E}_n, \alpha(\eta/s, \xi/s, \cdots))$$

- Anisotropic flow: $V_n = v_n e^{in\psi_n}$

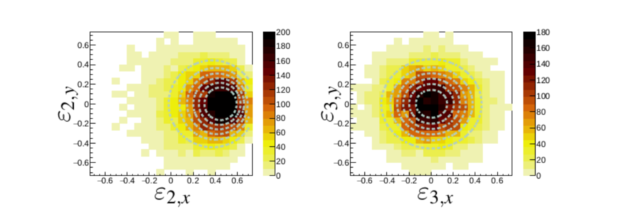
For $n = 2, 3$, The Hydrodynamic Response is Almost Linear

[Teaney, Yan, PRC, 2011], [Luzum, Ollitrault, . . .]

$$v_n \simeq \alpha_n \varepsilon_n$$



Different Analytical Distributions



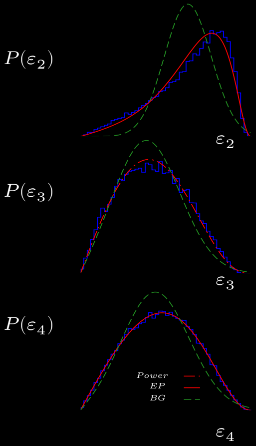
- Two Dimensional Gaussian Distribution

$$p(\varepsilon_{n,x},\varepsilon_{n,y}) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\Big[-\frac{(\varepsilon_{n,x}-\varepsilon_0)^2}{2\sigma_x^2}-\frac{\varepsilon_{n,y}^2}{2\sigma_y^2}\Big]$$

- Elliptic-Power Distribution **[Yan, Ollitrault, Poskanzer, PRC, 2014]**

$$p(\varepsilon_{n,x},\varepsilon_{n,y}) = \frac{\alpha}{\pi}(1-\varepsilon_0^2)^{\alpha+1/2}\frac{(1-\varepsilon_{n,x}^2-\varepsilon_{n,y}^2)^{\alpha-1}}{(1-\varepsilon_{n,x}\varepsilon_0)^{2\alpha+1}}$$

- Bessel-Gaussian **[Voloshin, Poskanzer,Tang, Wang, PLB, 2008]**
  $P(\varepsilon_n) = \frac{\varepsilon_n}{\sigma^2} e^{-\frac{\varepsilon_n^2+\varepsilon_0^2}{2\sigma^2}} I_0(\frac{\varepsilon_n \varepsilon_0}{\sigma^2})$
- Elliptic-Power **[Yan, Ollitrault, Poskanzer,PRC, 2014]**
  $P(\varepsilon_n) = 2\alpha\varepsilon_n(1-\varepsilon_n^2)^{\alpha-1}(1-\varepsilon_0^2)^{\alpha+1/2} \times \frac{1}{\pi}\int_0^{\pi}(1-\varepsilon_0\varepsilon_n\cos\varphi)^{-2\alpha-1}d\varphi.$
- Power (for odd n) **[Yan, Ollitrault, PRL, 2014]**
  $P(\varepsilon_n) = 2\alpha\varepsilon_n(1-\varepsilon_n^2)^{\alpha-1}$

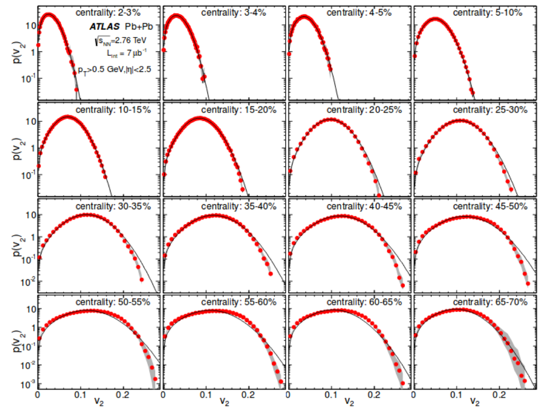


$75\% - 80\%$ *Centrality*

Bessel-Gaussian distribution

- Bessel-Gaussian

$$p(v_n) = (\frac{v_n}{\sigma^2}) I_0(\frac{v_n \bar{v}_n}{\sigma^2}) e^{-\frac{v_n^2+\bar{v}_n^2}{2\sigma^2}}$$



[ATLAS Collaboration,2013]

Standardized Cumulants of Flow Harmonic Fluctuations

[N.Abbasi,D.Allahbakhshi,A.Davody, S.F.Taghavi,PRC,2017]

- 2D Cumulants

$$\langle e^{x\lambda_x + y\lambda_y} \rangle = \exp\left[\sum_{m=n=0} \frac{\lambda_x^m \lambda_y^n}{m!n!} A_{mn}\right]$$

$$\hat{A}_{mn} \equiv A_{mn}/\sqrt{A_{20}A_{02}}$$

- Using relation between $c_n\{2k\}$ and A_{mn}

$$p_{odd}(v_n) = (\frac{v_n}{\sigma^2})e^{-\frac{v_n^2}{2\sigma^2}}\left[1 + \sum_{k=2} \frac{(-1)^k \Gamma_{2k-2}^{odd}}{k!} L_k(v_n^2/(2\sigma^2))\right]$$

Corrected Bessel-Gaussian distribution

[S.F.Taghavi, H.M, EPJC, 2018]

- Bessel-Gaussian

$$p(v_n) = (\frac{v_n}{\sigma^2}) I_0(\frac{v_n \bar{v}_n}{\sigma^2}) e^{-\frac{v_n^2+\bar{v}_n^2}{2\sigma^2}}$$

- Connecting $p(v_n)$ and $c_n\{2k\}$ to each other by employing a Gram–Charlier A series

$$p(v_n) = \sum_{k=0}^{\infty} (q_{2k} Q_{2k}) (\frac{v_n}{\sigma_2}) e^{-\frac{v_n^2+\bar{v}_n^2}{2\sigma^2}}$$

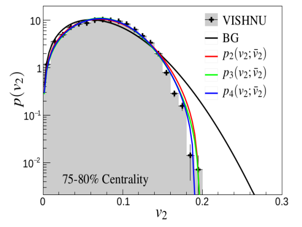
Considering $2\sigma^2 = v_n\{2\}^2 - \bar{v}_n^2$

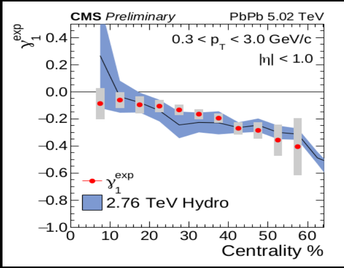
$$q_0 = 1$$

$$q_2 = 0$$

$$q_4 = -\frac{v_n\{4\}^4 - \bar{v}_n^4}{4(v_n\{2\}^2 - \bar{v}_n^2)^2}$$

$$q_6 = \frac{\bar{v}_n^6 - 3 v_n\{4\}^4 \bar{v}_n^2 + 2 v_n\{6\}^6}{18(v_n\{2\}^2 - \bar{v}_n^2)^3}$$





- Skewness **[Giacalone,Ollitrault,Yan, Noronha-Hostler,PRC,2016]**

  $$\gamma_1 \equiv -6\sqrt{2}v_2\{4\}^2 \frac{v_2\{4\}-v_2\{6\}}{(v_2\{2\}^2-v_2\{4\}^2)^{\frac{3}{2}}}.$$

  Gaussian approximation

  $$\bar{v}_2 = v_2\{4\}, 2\sigma_x^2 = v_2\{2\}^2 - v_2\{4\}^2$$

  In this case

  $q_4 = 0$

  $q_6 \propto \gamma_1$

  $q_8 \propto (v_2\{4\} - v_2\{6\}) - 11(v_2\{6\} - v_2\{8\})$

  $\vdots$

- Limits: $\bar{v}_n \to 0$

  [N.Abbasi,D.Allahbakhshi,A.Davody, S.F.Taghavi,PRC,2017]

Is there an unambiguous technique to find such radial flow distributions $p(v_n)$?

- Joint Generating Functions

$$\log G(\lambda) = \log\langle e^{\lambda z + \lambda^* z^*}\rangle = \sum_{k,l} \frac{\lambda^{*k}\lambda^l}{k!l!}\kappa\{k,l\} = K(\lambda)$$

Example: $1 + \sum_{n=1}^{\infty} \frac{\mu_n \lambda^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n \lambda^n}{n!}\right)$

$$K^{(n)}(\lambda)|_{\lambda=0} = (\log G(\lambda))^{(n)}|_{\lambda=0}$$

- Standard Method of Gram-Charlier Series

$$G(\lambda) = \exp[\sum_{n=3}^{\infty} \kappa_n \frac{(i\lambda)^n}{n!}]\exp[\kappa_1(i\lambda) + \kappa_2\frac{(i\lambda)^2}{2!}]$$

$$G(\lambda) \approx \left(1 + \sum_{n=3}^{\infty} \kappa_n \frac{(i\lambda)^n}{n!}\right) G_N(\lambda)$$

$$p(x) = \frac{1}{2\pi}\int d\lambda\, e^{-i\lambda x} G(t)$$

$$(i\lambda)^n G_N(\lambda) \iff (-D)^n G_N(x)$$

$$\log\langle e^{\lambda z+\lambda^* z^*}\rangle = \sum_{k,l} \frac{\lambda^{*k}\lambda^l}{k!l!}\kappa\{k,l\}$$

- Set $z \equiv V_n = v_{n,x} + iv_{n,y}$ and $\lambda \equiv (\lambda_x - i\lambda_y)/2$
- 2k-particle correlation functions:

$$c_n\{2\} = \langle v_n^2\rangle\,, \quad c_n\{4\} = \langle v_n^4\rangle - 2\langle v_n^2\rangle\,, \quad \cdots$$

- The form of $p_{odd}(v_n)$ can be found in terms of cumulants $c_n\{2k\}$!

$$p_{odd}(v_n) = (\frac{v_n}{\sigma^2})e^{-\frac{v_n^2}{2\sigma^2}}\left[1 + \sum_{k=2} \frac{(-1)^k\Gamma_{2k-2}^{odd}}{k!} L_k(v_n^2/(2\sigma^2))\right]$$

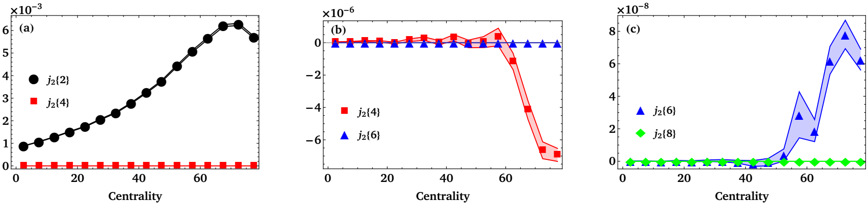
$$\log\langle e^{\lambda z+\lambda^* z^*}\rangle = \sum_{k,l} \frac{\lambda^{*k}\lambda^l}{k!l!}\kappa\{k,l\}$$

- Set $z \equiv W_n = (v_{n,x} - \bar{v}_n) + i v_{n,y}$ and $\lambda \equiv (\lambda_x - i\lambda_y)/2$
- New set of cumulants: $(w_n^2 = (v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2)$

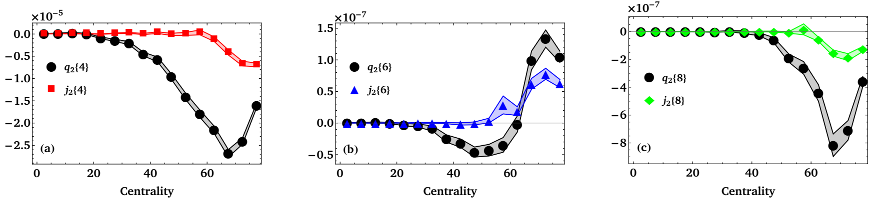
$$j_n\{2\} = \langle w_n^2 \rangle\,, \quad j_n\{4\} = \langle w_n^4 \rangle - 2\langle w_n^2 \rangle\,, \quad \cdots$$

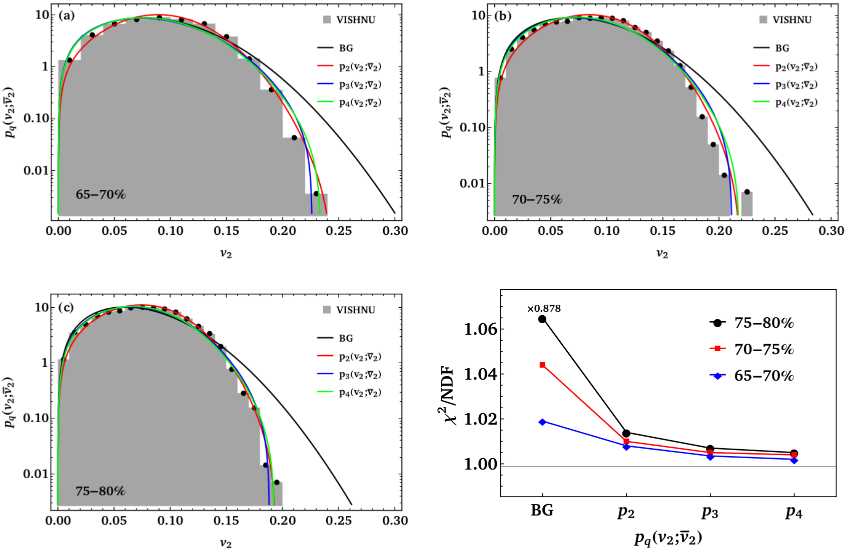
- The form of $p_{odd}(v_n)$ can be found in terms of cumulants $c_n\{2k\}$!

$$p_q(v_n;\bar{v}_n) = \mathcal{F}(v_n;\bar{v}_n)\sum_{k=0}^{q} \frac{(-1)^k\gamma_k}{k!}\left[\alpha'_k I_0\left(\frac{2v_n\bar{v}_n}{j_n\{2\}}\right) + \sum_{l=0}^{k}\beta_{kl} I_l\left(\frac{2v_n\bar{v}_n}{j_n\{2\}}\right)\right],$$



$$j_n\{2\} = q_n\{2\}, \text{ and } j_n\{2k\} = q_n\{2k\} + \dots, \text{ for } k \geq 2$$





(a)
VISHNU
BG
p2(v2;v̄2)
p3(v2;v̄2)
p4(v2;v̄2)
65–70%
(b)
70–75%
(c)
75–80%
×0.878
75–80%
70–75%
65–70%
χ2/NDF
BG
p2
p3
p4
pq(v2;v̄2)
v2

Joint Distribution of Flow Harmonics

- Experiments show that the event-plane correlations and symmetric cumulants are non-vanishing!

$$\mathcal{P}(v_1, v_2, ...) \neq \prod_n p(v_n)$$

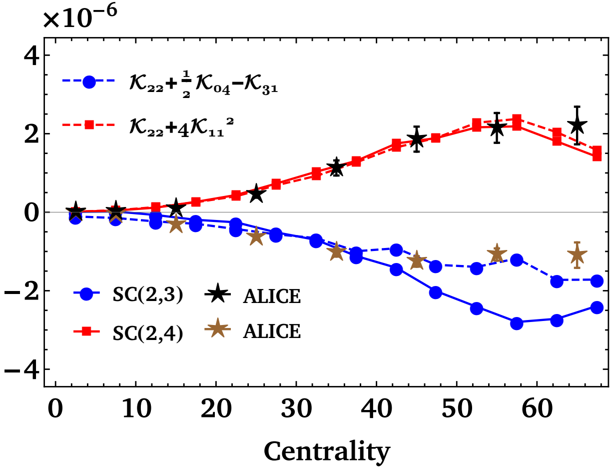
1. The correlations between flow harmonics
2. The event-by-event initial fluctuations
3. The correlations between different stages in heavy-ion collision processes

Can we find a joint flow distribution to interpret the most general form of the event-by-event fluctuations?

- Joint Generating function (two flow harmonics):

$$\langle e^{W_n\lambda_n+W_m\lambda_m}\rangle = \exp\left[\sum_{k,l=0}\frac{(\lambda_n)^k(\lambda_m)^l}{k!l!}\mathcal{K}_{kl}\right]$$

$$\begin{aligned}
&\mathcal{K}_{00}=\mathcal{K}_{10}=\mathcal{K}_{01}=0,\\
&\mathcal{K}_{11}=\langle v_n v_m\cos(\Psi_1-\Psi_2)\rangle-\bar{v}_n\bar{v}_m = Re[V_nV_m^*]-\bar{v}_n\bar{v}_m,\\
&\mathcal{K}_{20}=\langle v_n^2\rangle-\bar{v}_n^2=j_n\{2\},\\
&\mathcal{K}_{02}=\langle v_m^2\rangle-\bar{v}_m^2=j_m\{2\},\\
&\quad\vdots
\end{aligned}$$



×10⁻⁶
4
2
0
−2
−4
$\mathcal{K}_{22}+\frac{1}{2}\mathcal{K}_{04}-\mathcal{K}_{31}$
$\mathcal{K}_{22}+4\mathcal{K}_{11}{}^2$
SC(2,3)
SC(2,4)
ALICE
ALICE
0
10
20
30
40
50
60
Centrality

Applying Fourier transforming to both sides of joint generating functions

$$
\begin{aligned}
&\mathcal{P}(W_n, W_m) \\
&= \frac{1}{2\pi\Delta} \exp\left[\sum_{k+l\geq 3} \tilde{\mathcal{K}}_{kl} (\partial_n)^k (\partial_m)^l\right] \\
&\qquad \times \exp\left[-\frac{\tilde{\mathcal{K}}_{02} w_n^2 + \tilde{\mathcal{K}}_{20} w_m^2 - \tilde{\mathcal{K}}_{11}(w_{n,x} w_{m,x} + w_{n,y} w_{m,y})}{\Delta^2}\right] \\
&\approx \left[1 + \sum_{k+l\geq 3} \tilde{\mathcal{K}}_{kl} (\partial_n)^k (\partial_m)^l\right] \mathcal{N}(W_n, W_m)
\end{aligned}
$$

$$= 4\tilde{\mathcal{K}}_{20}\tilde{\mathcal{K}}_{02} - \tilde{\mathcal{K}}_{11}^2$$

Bivariate normal distribution

$\Longrightarrow \sigma_n^2 = 2\tilde{\mathcal{K}}_{20}, \ \sigma_m^2 = 2\tilde{\mathcal{K}}_{02},$ and $\rho_{nm} = \frac{\tilde{\mathcal{K}}_{11}}{2\sqrt{\tilde{\mathcal{K}}_{20}\tilde{\mathcal{K}}_{02}}}$

$$
\begin{aligned}
&\mathcal{P}(W_1, W_2, \ldots, W_n) \\
&\approx \left[1 + \sum_{k_1+\ldots+k_n\geq 3} \tilde{\mathcal{K}}_{k1\ldots kn} (\partial_1)^{k_1} \cdots (\partial_n)^{k_n}\right] \mathcal{N}(W_1, W_2, \ldots, W_n),
\end{aligned}
$$

[S.Voloshin and Y.Zhang, Z.Phys.C70, 1996]

$$\int dv_n dv_m \; \mathcal{P}(v_n; \bar{v}_n, v_m; \bar{v}_m)$$
$$\equiv \int v_n dv_n v_m dv_m \int d\Psi_m d\Psi_n d\Phi \; \frac{d\mathcal{P}(W_n, W_m)}{d\Psi_m d\Psi_n} \delta(\Phi - \Psi_m + \Psi_n).$$

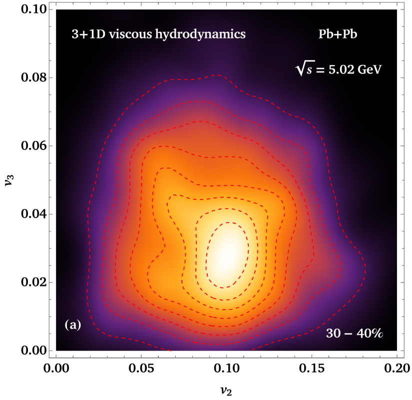
$$\int dv_n dv_m \; \mathcal{P}_1(v_n; \bar{v}_n, v_m; \bar{v}_m)$$
$$= \int dv_n dv_m d\Phi \; \chi_{mn} e^{\zeta_3 \cos\Phi} I_0 \left(\sqrt{\zeta_1^2 + \zeta_2^2 + 2\zeta_1\zeta_2 \cos(\Phi)} \right)$$

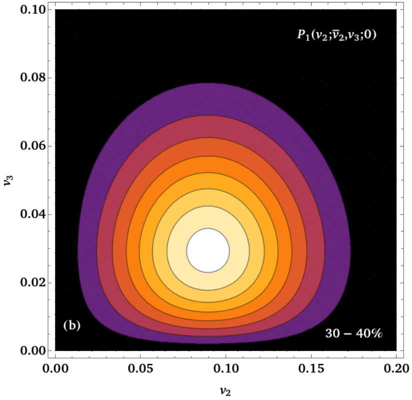
$$\chi_{mn} \equiv \frac{4v_n v_m}{\pi\Delta^2} \exp\left[-\frac{v_n^2 + \bar{v}_n^2}{\Delta^2/2\tilde{\mathcal{K}}_{02}} - \frac{v_m^2 + \bar{v}_m^2}{\Delta^2/2\tilde{\mathcal{K}}_{20}} + \frac{\bar{v}_n \bar{v}_m}{\Delta^2/2\tilde{\mathcal{K}}_{11}}\right],$$

$$\zeta_1 \equiv v_n\left(\frac{2\bar{v}_n}{\Delta^2/2\tilde{\mathcal{K}}_{02}} - \frac{\bar{v}_m}{\Delta^2/2\tilde{\mathcal{K}}_{11}}\right),$$

$$\zeta_2 \equiv v_m\left(\frac{2\bar{v}_m}{\Delta^2/2\tilde{\mathcal{K}}_{20}} - \frac{\bar{v}_n}{\Delta^2/2\tilde{\mathcal{K}}_{11}}\right),$$

$$\zeta_3 \equiv \frac{\bar{v}_n \bar{v}_m}{\Delta^2/2\tilde{\mathcal{K}}_{11}}.$$



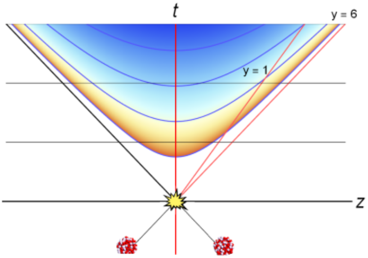


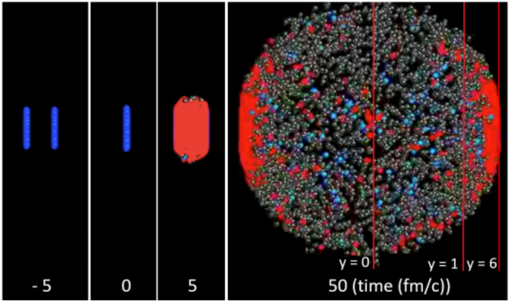
$$\mathcal{P}_1(v_2;\bar{v}_2,v_3;0)=\frac{4v_2v_3}{j_2\{2\}j_3\{2\}}\exp\left[-\frac{v_2^2+\bar{v}_2^2}{j_2\{2\}}-\frac{v_3^2}{j_3\{2\}}\right]I_0\left(\frac{2v_2\bar{v}_2}{j_2\{2\}}\right)$$

Conclusion

- I employed the relation between joint cumulant and moment generating function of flow harmonics to relate the radial flow distribution to cumulants by using the standard method of finding Gram-Charlier series.
- I have found a general form of radial flow distribution in terms of a general set of cumulants $j_n\{2k\}$.
- Also, I have obtained the odd flow distribution.
- I have obtained the joint distribution of flow harmonics and presented a general form for $\mathcal{P}(W_1, W_2, ..., W_n)$.
- I introduced new observables $\mathcal{K}_{nm}$ and showed that the experimental data for symmetric cumulants $SC(2,3)$ and $SC(2,4)$ can be explained by combinations of these observables.
- I also obtained the joint radial distribution of the two flow harmonics and showed that the first terms of this distribution for v_2 and v_3 can justify the simulation data.


THANK
YOU





- Momentum rapidity: $y = acosh(\gamma), \quad \gamma = 1/\sqrt{1 - v_z^2}$
- Space time rapidity: $y_s = atanh(z/t)$
- In a boost invariant flow: $v_z = z/t \implies y = y_s$