Strong Coupling Constants of Doubly Heavy $\boldsymbol{\Xi_{QQ'}}$ Baryons with $\boldsymbol{\pi}$ Mesons

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OUTLINE

- Introduction
- QCD Sum Rules
- Three Point and Light Cone QCDSR
- Multi-Heavy Hadrons
- Our Work
- Outlook

INTRODUCTION

It is based on QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_q \bar{\psi}_q (i \not{D} - m_q) \psi_q$$

Feynman Calculus works well at small $\alpha_s = g_s^2/4\pi$

Confinement ---> we have to work at scales of order $R_{hadr} \sim 1/\Lambda_{QCD}$

that is non-perturbative

INTRODUCTION

- One has to combine

Feynman Diagrams

(perturbative QCD)

+

wave functions or momentum distribution of quarks in hadrons

(non-perturbative QCD)

- One among many ways is QCD Sum Rules

QCD Sum Rules

- We Start With a Correlation Function

$$\Pi_{\mu\nu}(q) = i \int d^4x \, e^{iq\cdot x} \langle 0 | \, T\{j_\mu(x) j_\nu(0)\} \, |0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$$

- This is a 2-point CF : for Mass and decay constants

- We also use 3-point and Light Cone (LC) versions: for couplings and decay rates

QCD Sum Rules

- If $Q^2 \equiv -q^2 \gg \Lambda_{QCD}^2$, the CF turns into a genuine short-distance object we are looking for at $|x| \sim x_0 \sim 1/\sqrt{Q^2} \ll R_{hadr}$
- The quark-gluon interactions are suppressed due to asymptotic freedom
- In the 1st approximation virtual quarks can be considered as free
- In the case of heavy quarks is even simpler since $m_{b,c} \gg \Lambda_{QCD}$

QCD Sum Rules

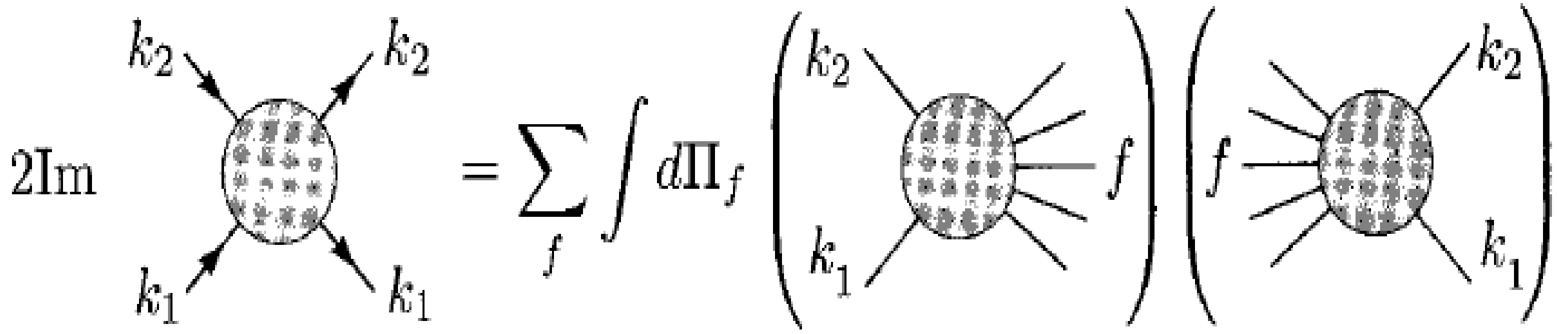
- **How the CF is related to physically observable hadrons?**

1. The invariant amp. $\Pi(q^2)$ is an analytic function of q^2
   for $q^2 > 0$ (time like)
   $q^2 < 0$ (space like) or even complex q^2
2. In $e^+ + e^- \to e^+ + e^-$ we have $E_{e^+} + E_{e^-} = \sqrt{q^2}$ ---> $q^2 > 0$
3. If we shift q^2 from (-) to (+), the distance x grows ---> hadrons

QCD Sum Rules

- **The hadronic content of CF at $q^2 > 0$**

1. Inserting the complete set of intermediate hadronic states
   $$\sum_{X_p} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left|X_p\right\rangle \left\langle X_p\right| = \mathbf{1}$$
2. Optical Theorem



QCD Sum Rules

- Unitarity relation

$$2\mathrm{Im}\Pi_{\mu\nu}(q^2) = \sum_X \langle 0|j_\mu|X\rangle\langle X|j_\nu|0\rangle d\tau_X (2\pi)^4\,\delta^4(q - q_X)$$

- Singling out the ground state

$$\frac{1}{\pi}\mathrm{Im}\Pi(q^2) = f_V^2\delta(q^2 - m_V^2) + \rho^h(q^2)\theta\left(q^2 - s_0^h\right)$$

where s_0^h is the threshold of the lowest continuum state

QCD Sum Rules

- **So the CF has a dual nature:**

1. At $q^2 \ll 0$: represents a short-distance $q\overline{q}$ fluctuations and can be treated perturbatively
2. At $q^2 > 0$: it has a decomposition in terms of hadronic observables

- The next step is to relate $\Pi(q^2)$ at $q^2 < 0$ to the hadronic sum.

QCD Sum Rules

- According to Schwartz reflection principle

$$\Pi(q^2 + i\epsilon) - \Pi(q^2 - i\epsilon) = 2i\,\mathrm{Im}\Pi(q^2)$$

we obtain the *Dispersion Relation (DR)*:

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds\, \frac{\mathrm{Im}\Pi(s)}{s - q^2 - i\epsilon}$$

- The CF is UV divergent ---> $\mathrm{Im}\Pi(s)$ does NOT vanish as $s \to \infty$
- To cure this we define $\overline{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$

QCD Sum Rules

- The DR is modified as

$$\overline{\Pi}(q^2) = \frac{q^2}{\pi}\int_{t_{min}}^{\infty} ds\, \frac{\mathrm{Im}\Pi(s)}{s(s-q^2)}$$

- Using the hadronic representation one finds

$$\Pi(q^2) = \frac{q^2 f_V^2}{m_V^2(m_V^2 - q^2)} + q^2 \int_{s_0^h}^{\infty} ds \frac{\rho^h(s)}{s(s-q^2)} + \Pi(0)$$

- Due to the gauge invariance : $\overline{\Pi}(0) = 0$

QCD Sum Rules

- The obtained sum rules is not yet very useful:

1. They are in general plagued by the presence of unknown subtraction terms.
2. Little is known about the spectral function $\rho^h(s)$ of excited and continuum states.
3. The integrals are usually diverges (if not always!)

- To remove the divergences and improve the situation we use the Borel transformation

QCD Sum Rules

- **The Borel transformation:**

1. It is a standard mathematical technique which can be used to improve the radius of convergence of a function $\Pi(x)$
2. Suppose we have the Taylor series of $\Pi(x)$ as $\Pi(x) = \sum_{n=0}^{\infty} f_n x^n$
3. We define the Borel transformation $\Pi(x)$ as $\widetilde{\Pi}\,(x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n$
4. On can show that $\Pi(x) = \frac{1}{x}\int_0^{\infty} dz\ \widetilde{\Pi}\,(z) e^{-z/x}$
5. And $\widetilde{\Pi}\,(M^2) = \mathcal{B}\Pi(q^2) = \lim_{\substack{-q^2,n\to\infty \\ -q^2/n=M^2}} \frac{(-q^2)^{(n+1)}}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)$

QCD Sum Rules

- Some important Borel transformations

1. $\mathcal{B}_{M^2}(q^2)^k = 0$

2. $\mathcal{B}_{M^2}\left(\frac{1}{(m^2-q^2)^k}\right) = \frac{1}{(k-1)!}\frac{\exp(-m^2/M^2)}{M^{2(k-1)}}$

3. $\mathcal{B}_{M^2}\left(e^{-\alpha q^2}\right) = \delta(1-\alpha M^2)$

QCD Sum Rules

- A more convenient form of sum rules is obtained :

$$\Pi(M^2) = f_V^2 e^{-m_V^2/M^2} + \int_{s_0^h}^{\infty} ds\ \rho^h(s) e^{-s/M^2}$$

- The Borel tr. removes subtraction terms in dispersion relation.
- It exponentially suppresses the contributions from excited resonances and continuum states heavier than meson V.

QCD Sum Rules

- **Calculating the perturbative part of CF in QCD**

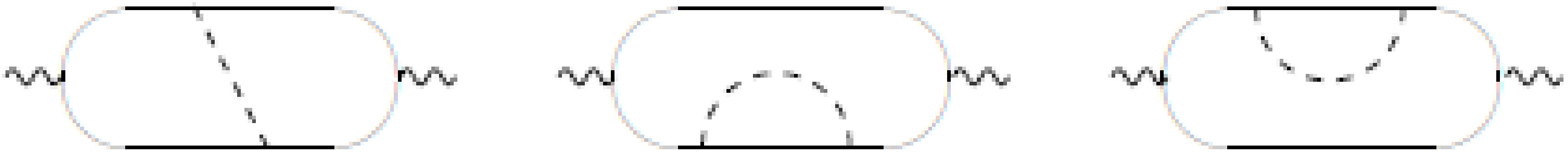
1. At $Q^2 = -q^2 \gg 0$, the CF $\Pi_{\mu\nu}$ can be approximated by free-quark loop diagrams

with free-quark propagator

$$S_0^{ij}(x,y) = -i\langle 0|T\{\psi^i(x)\overline{\psi}^j(y)\}|0\rangle = \delta^{ij}\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\not{p}+m}{p^2-m^2}$$

QCD Sum Rules

2. To improve the free-quark approximation, one has to calculate the $O(\alpha_s)$ perturbative corrections to the following diagrams

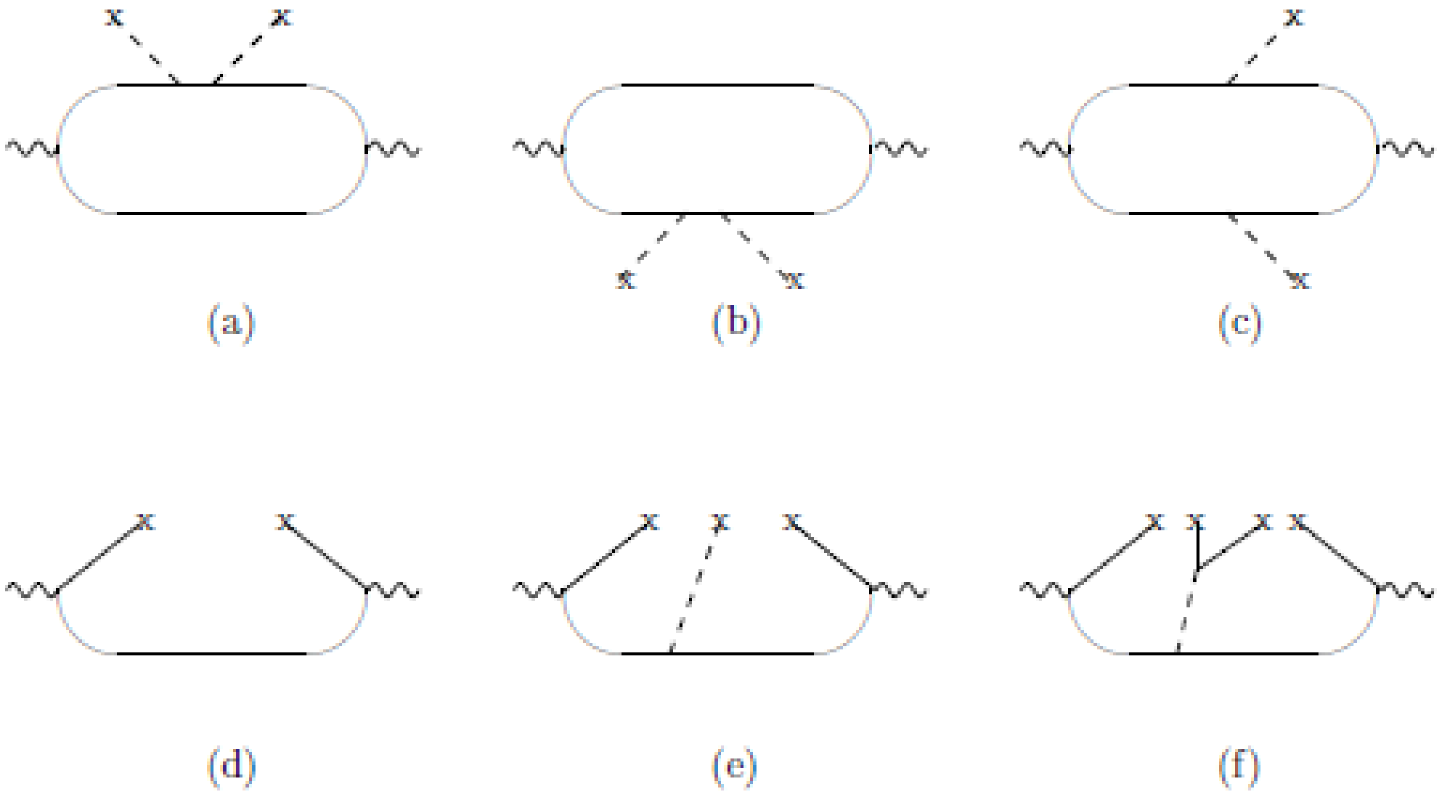


QCD Sum Rules

- The perturbative part of CF does not contain all important contributions!
- The complete calculation has to include the effects due to the fields of soft gluons and quarks populating the vacuum of QCD.
- Vacuum fluctuations in QCD are due to the complicated nonlinear nature of the QCD Lagrangian.
- Various non-perturbative approaches (instantons, lattice, …) indicates that $\Lambda_{vac} \sim \Lambda_{QCD}$

QCD Sum Rules

- The non-perturbative effects



QCD Sum Rules

OPE

- To incorporate both short- and long-distance effects and separate them we use the framework of generalized Wilson OPE
- One has to expand the product of two currents in a series of local operators:

$$i\int d^4x\, e^{iq\cdot x}\, T\{\bar{\psi}(x)\gamma_\mu\psi(x), \bar{\psi}(0)\gamma_\nu\psi(0)\}$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\sum_d C_d(q^2) O_d\,,$$

so that

$$\Pi(q^2) = \sum_d C_d(q^2)\langle 0|\, O_d\, |0\rangle$$

QCD Sum Rules

- The list of operators with low dimensions entering OPE :

$$O_3 = \bar{\psi}\psi \qquad O_4 = G^a_{\mu\nu}G^{a\mu\nu} \quad O_5 = \bar{\psi}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^{a\mu\nu}\psi\,,$$

$$O_6^\psi = (\bar{\psi}\Gamma_r\psi)(\bar{\psi}\Gamma_s\psi) \qquad O_6^G = f_{abc}G^a_{\mu\nu}G^{b\,\nu}_{\sigma}G^{c\sigma\mu}$$

- Expanding quarks wave function we have

$$\psi(x) = \psi(0) + x^\rho \overrightarrow{D}_\rho\,\psi(0) + ..\,,$$

$$\bar{\psi}(x) = \bar{\psi}(0) + \bar{\psi}(0)\overleftarrow{D}_\rho\,x^\rho + ..\,,$$

QCD Sum Rules

The full quark propagator would be

$$S_q(x) = \frac{i\not{x}}{2\pi^2 x^4} - \frac{m_q}{4\pi^2 x^2} - \frac{\langle \bar{q}q \rangle}{12}\left(1 - i\frac{m_q}{4}\not{x}\right)$$
$$-\frac{x^2}{192} m_0^2 \langle \bar{q}q \rangle \left(1 - i\frac{m_q}{6}\not{x}\right)$$
$$-ig_s \int_0^1 \mathrm{d}u \left[\frac{\not{x}}{16\pi^2 x^2} G_{\mu\nu}(ux)\sigma_{\mu\nu}\right.$$
$$-\frac{i}{4\pi^2 x^2} ux^\mu G_{\mu\nu}(ux)\gamma^\nu$$
$$\left.-i\frac{m_q}{32\pi^2} G_{\mu\nu}(ux)\sigma^{\mu\nu}\left(\ln\left(\frac{-x^2\Lambda^2}{4}\right) + 2\gamma_E\right)\right]$$

$$S_Q(x) = \frac{m_Q^2}{4\pi^2}\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}} - i\frac{m_Q^2\not{x}}{4\pi^2 x^2} K_2\left(m_Q\sqrt{-x^2}\right)$$
$$-ig_s \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx} \int_0^1 \mathrm{d}u$$
$$\times\left[\frac{\not{k} + m_Q}{2(m_Q^2 - k^2)^2} G^{\mu\nu}(ux)\sigma_{\mu\nu}\right.$$
$$\left.+\frac{u}{m_Q^2 - k^2} x_\mu G^{\mu\nu}(ux)\gamma_\nu\right],$$

QCD Sum Rules

- **Ioffe Current and sum rules for baryons**

1. One needs a baryon current having the same quantum number as the given baryon
2. For proton

$$J^N(x) = \epsilon_{abc}(u^{aT}(x)C\gamma_\mu u^b(x))\gamma_5\gamma^\mu d^c(x)$$

or

$$J'^N(x) = \epsilon_{abc}(u^{aT}(x)C\sigma_{\mu\nu} u^b(x))\gamma_5\sigma^{\mu\nu} d^c(x)$$

QCD Sum Rules

- With a little effort one defines the interpolating currents for L=0 baryonic octet

$$J^{\Sigma}(x) = \epsilon_{abc}(u^{aT}(x)C\gamma_{\mu}u^{b}(x))\gamma_5\gamma^{\mu}s^{c}(x)\,,$$

$$J^{\Xi}(x) = -\epsilon_{abc}(s^{aT}(x)C\gamma_{\mu}s^{b}(x))\gamma_5\gamma^{\mu}u^{c}(x)$$

$$J^{\Lambda}(x) = \sqrt{\frac{2}{3}}\epsilon_{abc}\Big[(u^{aT}(x)C\gamma_{\mu}s^{b}(x))\gamma_5\gamma^{\mu}d^{c}(x) - (d^{aT}(x)C\gamma_{\mu}s^{b}(x))\gamma_5\gamma^{\mu}u^{c}(x)\Big]$$

QCD Sum Rules

- And for the L=0 decuplet:

$$J_\mu^{\Delta}(x) \;=\; \epsilon_{abc}(u^{aT}(x)\mathcal{C}\gamma_\mu u^b(x))u^c(x)$$

$$J_\mu^{\Sigma^*}(x) \;=\; \sqrt{\frac{1}{3}}\epsilon_{abc}\Big[2(u^{aT}(x)\mathcal{C}\gamma_\mu s^b(x))u^c(x) + (u^{aT}(x)\mathcal{C}\gamma_\mu u^b(x))s^c(x)\Big]$$

$$J_\mu^{\Xi^*}(x) \;=\; \sqrt{\frac{1}{3}}\epsilon_{abc}\Big[2(s^{aT}(x)\mathcal{C}\gamma_\mu u^b(x))s^c(x) + (s^{aT}(x)\mathcal{C}\gamma_\mu s^b(x))u^c(x)\Big]$$

$$J_\mu^{\Omega}(x) \;=\; \epsilon_{abc}(s^{aT}(x)\mathcal{C}\gamma_\mu s^b(x))s^c(x)$$

Three-point Sum Rules

- To calculate hadronic matrix elements of EM and weak
- Form factors, decay amplitudes, strong couplings
- One starts from three-point CF and uses double dispersion relation
- Extensively uses for both light and heavy quarks
- The generic CF would be

$$T_{\mu\nu\lambda}(p,p') = (i)^2 \int d^4x\, d^4y\, e^{i(p'\cdot x - p\cdot y)} \langle 0|T\{j_\mu^{(\pi)\dagger}(x) j_\lambda^{em}(0) j_\nu^{(\pi)}(y)\}|0\rangle$$

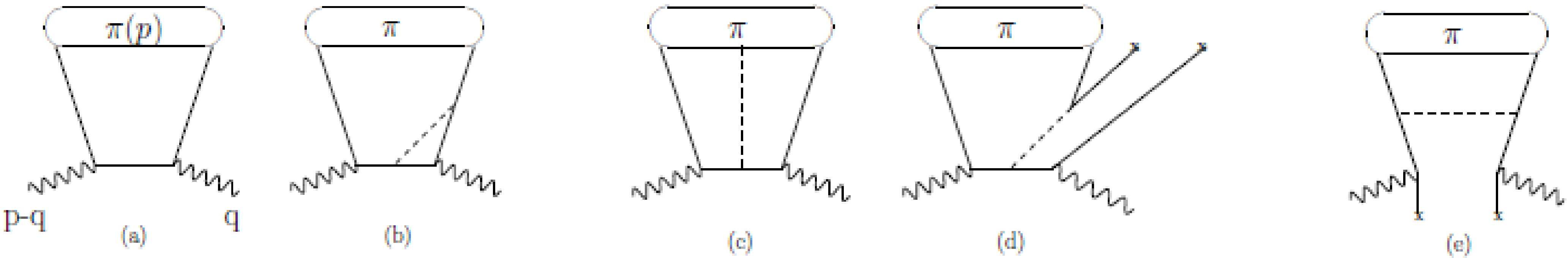
Light-Cone Sum Rules

- The basic idea is to OPE near the light cone
- Involves a partial resummation of local operators (in terms of twists) and avoid certain irregularities of the truncated OPE in 3-point SR.
- While SVZ sum rules employs vacuum-to-vacuum CF, LCSR sandwiches the two interpolating currents between vacuum and on-shell state

$$F_{\mu\nu}(p,q) = i\int d^4x\, e^{-iq\cdot x}\langle\pi^0(p)\,|\,T\{j_\mu^{em}(x)j_\nu^{em}(0)\}\,|\,0\rangle$$

Light-Cone Sum Rules

- Light-cone expansion of the CF



Multi-Heavy Hadrons

- Many ground and higher states at different light and heavy channels have been discovered by the experiments.
- Roughly all light and heavy mesons predicted by the quark model have been observed.
- All light and single charmed ground state baryons together with some excited states have been detected by different experiments as well.
- In the case of heavy b-baryons, except for the $\Omega_b{}^*$, all single heavy baryons have been seen.
- For standard baryons with 2 or 3 heavy quarks, only Ξ_{cc} has been discovered (SELEX 2002, 2005).
- The LHCb (2017) has reported the observation of $\Xi_{cc}{}^{++}$ via $\Lambda_c{}^+K^-\pi^+\pi$ decay mode with mass $3621.40 \pm 0.72(\text{stat.}) \pm 0.27(\text{syst.}) \pm 0.14\left(\Lambda_c{}^+\right)$.

Multi-Heavy Hadrons

- Exotic hadrons are not forbidden by QCD and predicted by Jaffe.
- Belle (2003) discovered $X(3872)$, LHCb (2015) discovered ${P_c}^+(4380)$ and ${P_c}^+(4450)$.
- The nature and structure of most exotics states remain unclear.
- Up to now, studies of hadrons with light and one heavy quark are well covered.
- The investigation of hadrons with two or more heavy quarks are still rare.
- To study them, we should develop the mathematics of light and heavy systems with more quarks.
- Baryons with full b-quarks and without light ones are called beauty-full states.

Multi-Heavy Hadrons

- For multi-heavy hadrons, especially when the heavy quarks are same, it leads to a well-known problem and we end up with indeterminate result in the calculations of continuum subtractions.
- We proposed a solution that removes the divergences for all the multi-heavy states up to beauty-full penta-quark hadrons.
- The phrase beautiful mathematics insists that the results are finite and do not include any divergences for all multi-heavy states.
- Here we present the results from light systems up to 5 heavy ones.

Multi-Heavy Hadrons

- The CF:

$$\Pi = i \int \mathrm{d}^4 x e^{ipx} \langle \mathcal{P}(q) | \mathcal{T} \{ \eta(x) \bar{\eta}(0) \} | 0 \rangle$$

- The interpolating currents:

$$\eta^S = \frac{1}{\sqrt{2}} \epsilon_{abc} \Big\{ (Q^{aT} C q^b) \gamma_5 Q'^c + (Q'^{aT} C q^b) \gamma_5 Q^c$$
$$+ \beta (Q^{aT} C \gamma_5 q^b) Q'^c + \beta (Q'^{aT} C \gamma_5 q^b) Q^c \Big\},$$
$$\eta^A = \frac{1}{\sqrt{6}} \epsilon_{abc} \Big\{ 2 (Q^{aT} C Q'^b) \gamma_5 q^c + (Q^{aT} C q^b) \gamma_5 Q'^c$$
$$- (Q'^{aT} C q^b) \gamma_5 Q^c + 2\beta (Q^{aT} C \gamma_5 Q'^b) q^c$$
$$+ \beta (Q^{aT} C \gamma_5 q^b) Q'^c - \beta (Q'^{aT} C \gamma_5 q^b) Q^c \Big\},$$

The Hadronic Part

$$\Pi^{\text{Phys.}}(p,q) = \frac{\langle 0|\eta|B_2(p,r)\rangle\langle B_2(p,r)\mathcal{P}(q)|B_1(p+q,s)\rangle\langle B_1(p+q,s)|\bar{\eta}|0\rangle}{(p^2-m_1^2)[(p+q)^2-m_2^2]} + \cdots$$

$$\langle 0|\eta|B_i(p,s)\rangle \;\; = \;\; \lambda_{B_i} u(p,s),$$

$$\langle B_2(p,r)\mathcal{P}(q)|B_1(p+q,s)\rangle \;\; = \;\; g_{B_1 B_2 \mathcal{P}}\bar{u}(p,r)\gamma_5 u(p+q,s) \;,$$

The Hadronic Part

$$\Pi^{\text{Phys.}}(p,q) = \frac{g_{B_1 B_2 \mathcal{P}} \lambda_{B_1} \lambda_{B_2}}{(p^2 - m_{B_2}^2)[(p+q)^2 - m_{B_1}^2]} [\not{q}\not{p}\gamma_5 + \cdots\ ] + \cdots$$

$$\begin{aligned} \mathcal{B}_{p_1}(M_1^2)\mathcal{B}_{p_2}(M_2^2)\Pi^{\text{Phys.}}(p,q) &\equiv \Pi^{\text{Phys.}}(M^2) \\ &= g_{B_1 B_2 \mathcal{P}} \lambda_{B_1} \lambda_{B_2} e^{-m_{B_1}^2/M_1^2} e^{-m_{B_2}^2/M_2^2} \not{q}\not{p}\gamma_5\ + \cdots \end{aligned}$$

The QCD Side

- The selected Structure is

$$\Pi^{\mathrm{QCD}}(p,q) = \Pi(p,q)\not{q}\not{p}\gamma_5,$$

- After inserting the interpolating currents into the CF one gets the QCD side as

Multi-Heavy Hadrons

- for symmetric part we get

$$
\begin{aligned}
\Pi^{\rm QCD}_{(\mathcal{S})\rho\sigma}(p,q) &= \frac{i}{2}\epsilon_{abc}\epsilon_{a'b'c'}\int d^4x e^{iq.x}\langle\mathcal{P}(q)|\bar{q}^{c'}_{\alpha}(0)q^{c}_{\beta}(x)|0\rangle\Bigg\{\Bigg[\Big(\tilde{S}^{aa'}_{Q}(x)\Big)_{\alpha\beta}\Big(\gamma_5 S^{bb'}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} \\
&+ \Big(\gamma_5 S^{bb'}_{Q'}(x)C\Big)_{\rho\alpha}\Big(CS^{aa'}_{Q}(x)\gamma_5\Big)_{\beta\sigma} + t\Big\{\Big(\gamma_5\tilde{S}^{aa'}_{Q}(x)\Big)_{\alpha\beta}\Big(\gamma_5 S^{bb'}_{Q'}(x)\Big)_{\rho\sigma} \\
&+ \Big(\tilde{S}^{aa'}_{Q}(x)\gamma_5\Big)_{\alpha\beta}\Big(S^{bb'}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} + \Big(\gamma_5 S^{bb'}_{Q'}(x)C\gamma_5\Big)_{\rho\alpha}\Big(CS^{aa'}_{Q}(x)\Big)_{\beta\sigma} \\
&- \Big(S^{bb'}_{Q'}(x)C\Big)_{\rho\alpha}\Big(\gamma_5 CS^{aa'}_{Q}(x)\gamma_5\Big)_{\beta\sigma}\Big\} + t^2\Big\{\Big(\gamma_5\tilde{S}^{aa'}_{Q}(x)\gamma_5\Big)_{\alpha\beta}\Big(S^{bb'}_{Q'}(x)\Big)_{\rho\sigma} \\
&- \Big(S^{bb'}_{Q'}(x)C\gamma_5\Big)_{\rho\alpha}\Big(\gamma_5 CS^{aa'}_{Q}(x)\Big)_{\beta\sigma}\Big\}\Bigg] + \Bigg(Q\longleftrightarrow Q'\Bigg)\Bigg\}, \qquad (10)
\end{aligned}
$$

Antisymmetric part:

$$
\begin{aligned}
\Pi^{\rm QCD}_{(\mathcal{A})\rho\sigma}(p,q) &= \frac{i}{6}\epsilon_{abc}\epsilon_{a'b'c'}\int d^4x e^{iq.x}\langle\mathcal{P}(q)|\bar{q}^{c'}_{\alpha}(0)q^{c}_{\beta}(x)|0\rangle\Bigg\{4\mathrm{Tr}\left[\tilde{S}^{aa'}_{Q}(x)S^{bb'}_{Q'}(x)\right]\gamma^5_{\alpha\sigma}\gamma^5_{\rho\beta} \\
&- 2\Big(\tilde{S}^{aa'}_{Q}(x)S^{bb'}_{Q'}(x)\gamma^5\Big)_{\alpha\sigma}\gamma^5_{\rho\beta} - 2\Big(\gamma^5 S^{bb'}_{Q'}(x)\tilde{S}^{aa'}_{Q}(x)\Big)_{\rho\beta}\gamma^5_{\alpha\sigma} \\
&- 2\Big(\tilde{S}^{bb'}_{Q'}(x)S^{aa'}_{Q}(x)\gamma^5\Big)_{\alpha\sigma}\gamma^5_{\rho\beta} - 2\Big(\gamma^5 S^{aa'}_{Q}(x)\tilde{S}^{bb'}_{Q'}(x)\Big)_{\rho\beta}\gamma^5_{\alpha\sigma} \\
&+ \Big(\tilde{S}^{aa'}_{Q}(x)\Big)_{\alpha\beta}\Big(\gamma^5 S^{bb'}_{Q'}(x)\gamma^5\Big)_{\rho\sigma} + \Big(\tilde{S}^{bb'}_{Q'}(x)\Big)_{\alpha\beta}\Big(\gamma^5 S^{aa'}_{Q}(x)\gamma^5\Big)_{\rho\sigma} \\
&+ \Big(\gamma^5 S^{bb'}_{Q'}(x)C\Big)_{\rho\alpha}\Big(CS^{aa'}_{Q}(x)\gamma^5\Big)_{\beta\sigma} + \Big(\gamma^5 S^{aa'}_{Q}(x)C\Big)_{\rho\alpha}\Big(CS^{bb'}_{Q'}(x)\gamma^5\Big)_{\beta\sigma} \\
&+ t\Bigg[4\mathrm{Tr}\left[\tilde{S}^{aa'}_{Q}(x)S^{bb'}_{Q'}(x)\gamma^5\right]\gamma^5_{\rho\beta}\delta_{\alpha\sigma} + 4\mathrm{Tr}\left[S^{bb'}_{Q'}(x)\tilde{S}^{aa'}_{Q}(x)\gamma^5\right]\gamma^5_{\alpha\sigma}\delta_{\rho\beta} \\
&+ 2\Big(\tilde{S}^{bb'}_{Q'}(x)\gamma^5 S^{aa'}_{Q}(x)C\gamma^5\Big)_{\alpha\sigma}\delta_{\beta\rho} - 2\Big(S^{aa'}_{Q}(x)\tilde{S}^{bb'}_{Q'}(x)\gamma^5\Big)_{\beta\rho}\gamma^5_{\alpha\sigma} \\
&- 2\Big(\gamma^5\tilde{S}^{aa'}_{Q}(x)S^{bb'}_{Q'}(x)\Big)_{\alpha\sigma}\gamma^5_{\rho\beta} - 2\Big(\gamma^5 S^{bb'}_{Q'}(x)\gamma^5\tilde{S}^{aa'}_{Q}(x)\Big)_{\rho\beta}\delta_{\sigma\alpha} \\
&- 2\Big(\gamma^5\tilde{S}^{bb'}_{Q'}(x)S^{aa'}_{Q}(x)\Big)_{\alpha\sigma}\gamma^5_{\rho\beta} - 2\Big(\gamma^5 S^{aa'}_{Q}(x)\gamma^5\tilde{S}^{bb'}_{Q'}(x)\Big)_{\rho\beta}\delta_{\alpha\sigma} \\
&- 2\Big(\tilde{S}^{aa'}_{Q}(x)\gamma^5 S^{bb'}_{Q'}(x)\gamma^5\Big)_{\alpha\sigma}\delta_{\rho\beta} - 2\Big(S^{bb'}_{Q'}(x)\tilde{S}^{aa'}_{Q}(x)\gamma^5\Big)_{\rho\beta}\gamma^5_{\alpha\sigma} \\
&+ \Big(\gamma^5\tilde{S}^{aa'}_{Q}(x)\Big)_{\alpha\beta}\Big(\gamma^5 S^{bb'}_{Q'}(x)\Big)_{\rho\sigma} + \Big(\gamma^5 S^{bb'}_{Q'}(x)C\gamma^5\Big)_{\rho\alpha}\Big(CS^{aa'}_{Q}(x)\Big)_{\beta\sigma} \\
&+ \Big(\tilde{S}^{aa'}_{Q}(x)\gamma^5\Big)_{\alpha\beta}\Big(S^{bb'}_{Q'}(x)\gamma^5\Big)_{\rho\sigma} + \Big(\gamma^5 CS^{aa'}_{Q}(x)\gamma^5\Big)_{\beta\sigma}\Big(S^{bb'}_{Q'}(x)C\Big)_{\rho\alpha} \\
&+ \Big(S^{aa'}_{Q}(x)C\Big)_{\rho\alpha}\Big(\gamma^5 CS^{bb'}_{Q'}(x)\gamma^5\Big)_{\beta\sigma} + \Big(\tilde{S}^{bb'}_{Q'}(x)\gamma^5\Big)_{\alpha\beta}\Big(S^{aa'}_{Q}(x)\gamma^5\Big)_{\rho\sigma} \\
&+ \Big(\gamma^5 S^{aa'}_{Q}(x)C\gamma^5\Big)_{\rho\alpha}\Big(CS^{bb'}_{Q'}(x)\Big)_{\beta\sigma} + \Big(\gamma^5\tilde{S}^{bb'}_{Q'}(x)\Big)_{\alpha\beta}\Big(\gamma^5 S^{aa'}_{Q}(x)\Big)_{\rho\sigma}\Bigg] \\
&+ t^2\Bigg[4\mathrm{Tr}\left[\tilde{S}^{aa'}_{Q}(x)\gamma^5 S^{bb'}_{Q'}(x)\gamma^5\right]\delta_{\alpha\sigma}\delta_{\beta\rho} - 2\Big(\gamma^5\tilde{S}^{aa'}_{Q}(x)\gamma^5 S^{bb'}_{Q'}(x)\Big)_{\alpha\sigma}\delta_{\rho\beta} \\
&+ 2\Big(S^{aa'}_{Q}(x)\gamma^5\tilde{S}^{bb'}_{Q'}(x)\gamma^5\Big)_{\rho\beta}\delta_{\alpha\sigma} - 2\Big(S^{bb'}_{Q'}(x)\gamma^5\tilde{S}^{aa'}_{Q}(x)\gamma^5\Big)_{\rho\beta}\delta_{\alpha\sigma} \\
&+ 2\Big(\gamma^5\tilde{S}^{bb'}_{Q'}(x)\gamma^5 S^{aa'}_{Q}(x)\Big)_{\alpha\sigma}\delta_{\beta\rho} + \Big(S^{bb'}_{Q'}(x)\Big)_{\rho\sigma}\Big(\gamma^5\tilde{S}^{aa'}_{Q}(x)\gamma^5\Big)_{\alpha\beta} \\
&+ \Big(S^{aa'}_{Q}(x)\Big)_{\rho\sigma}\Big(\gamma^5\tilde{S}^{bb'}_{Q'}(x)\gamma^5\Big)_{\alpha\beta} + \Big(S^{bb'}_{Q'}(x)C\gamma^5\Big)_{\sigma\alpha}\Big(\gamma^5 CS^{aa'}_{Q}(x)\Big)_{\beta\rho} \\
&+ \Big(S^{aa'}_{Q}(x)C\gamma^5\Big)_{\rho\alpha}\Big(\gamma^5 CS^{bb'}_{Q'}(x)\Big)_{\beta\sigma}\Bigg]\Bigg\}. \qquad (11)
\end{aligned}
$$

The Divergence Problem

- The generic form for the CF of two-heavy system:

$$T_2 = \int_0^1 \mathrm{d}u \int \mathrm{d}^4x \, e^{iP\cdot x} \, A(u)$$

$$\times \frac{K_\nu(m_{1Q}\sqrt{-x^2})\, K_\mu(m_{2Q}\sqrt{-x^2})}{(\sqrt{-x^2})^n}$$

- This is where we run into divergences in the spectral densities!
- Even multiple Borel transformation cannot remove these divergences.

The Divergence Problem

- REASON:

  The Borel tr. Is not strong enough to remove the residual divergences.

- SOLUTION:

  We have to someway enlarge the radius of the convergence of the CF

- HOW?
- Normally people use the exponential representation of the Bessel function. Maybe the problem is that we don't use the proper representation !

The Divergence Problem

- Instead of the exponential representation

$$\frac{K_\nu(m_Q\sqrt{-x^2})}{(\sqrt{-x^2})^\nu} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t^{\nu+1}} \exp\left[-\frac{m_Q}{2}\left(t - \frac{x^2}{t}\right)\right]$$

we use the cosine representation as

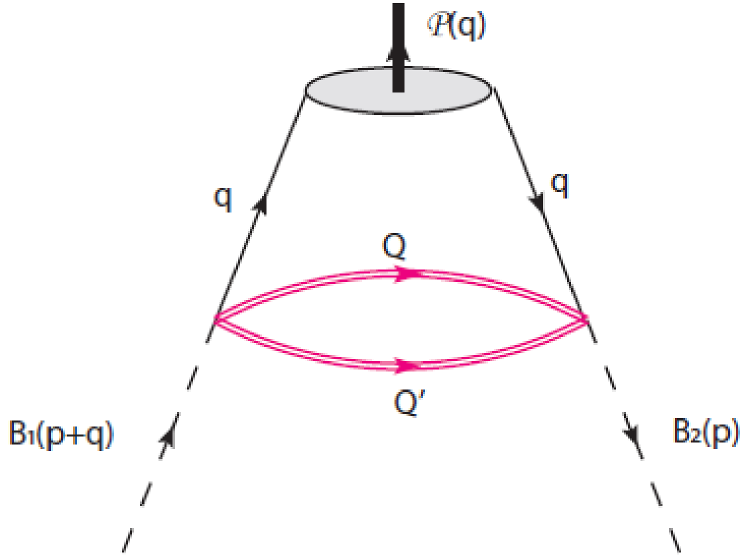
$$K_\nu\left(m_Q\sqrt{-x^2}\right) = \frac{\Gamma(\nu+1/2)\,2^\nu}{\sqrt{\pi}m_Q^\nu}\int_0^\infty \mathrm{d}t\cos(m_Q t) \times \frac{(\sqrt{-x^2})^\nu}{(t^2 - x^2)^{\nu+1/2}},$$

The full heavy quark propagator

$$
\begin{aligned}
S_Q^{aa'}(x) \;=\;& \frac{m_Q^2}{4\pi^2}\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}}\delta^{aa'} - i\frac{m_Q^2 \not{x}}{4\pi^2 x^2}K_2(m_Q\sqrt{-x^2})\delta^{aa'} \\
-\;& i g_s \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int_0^1 du \Bigg[\frac{\not{k} + m_Q}{2(m_Q^2 - k^2)^2}\sigma^{\mu\nu} G_{\mu\nu}^{aa'}(ux) \\
+\;& \frac{u}{m_Q^2 - k^2} x^\mu \gamma^\nu G_{\mu\nu}^{aa'}(ux) \Bigg] + \cdots .
\end{aligned}
$$

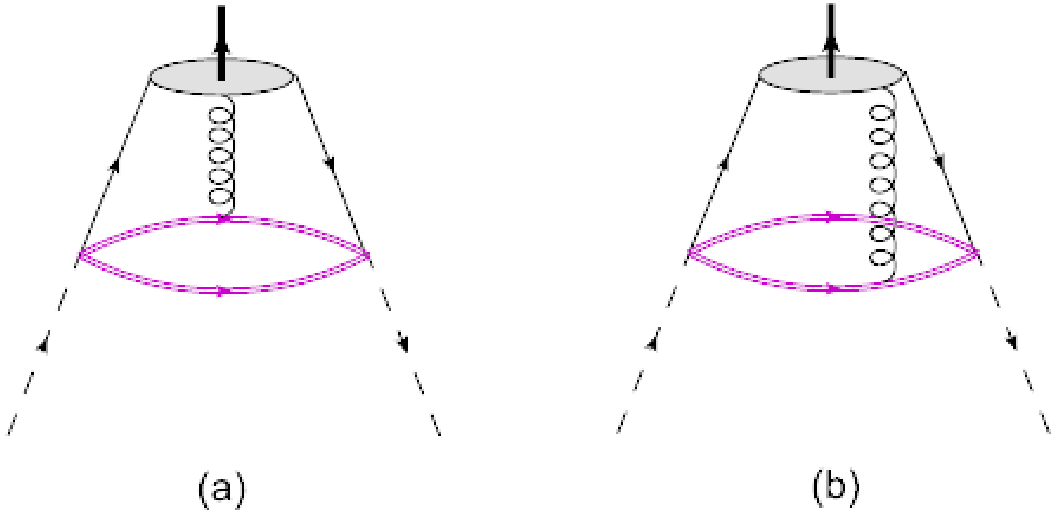
$G_{\mu\nu}^{aa'} \equiv G_{\mu\nu}^A t_A^{aa'}$ $\qquad t_A = \lambda_A/2$ $\qquad A = 1, 2 \ldots 8$

$$S_Q^{(\mathrm{pert.})}(x) \;=\; \frac{m_Q^2}{4\pi^2}\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}} - i\frac{m_Q^2 \not{x}}{4\pi^2 x^2}K_2(m_Q\sqrt{-x^2}).$$



$$S_Q^{aa'(non-p.)}(x) = -ig_s \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int_0^1 du G_{\mu\nu}^{aa'}(ux) \Delta_Q^{\mu\nu}(x),$$

$$\Delta_Q^{\mu\nu}(x) = \frac{1}{2(m_Q^2 - k^2)^2} \Big[(\not{k} + m_Q)\sigma^{\mu\nu} + 2u(m_Q^2 - k^2) x^\mu \gamma^\nu \Big].$$



Fiertz Identities

$$\bar{q}^{c'}_{\alpha} q^{c}_{\beta} \to -\frac{1}{12}(\Gamma_J)_{\beta\alpha}\delta^{cc'}\bar{q}\Gamma_J q,$$

$$\bar{q}^{c'}_{\alpha} q^{c}_{\beta} G^{A}_{\lambda\theta} \to -\frac{1}{4}\frac{1}{4}\left(\frac{\lambda^A}{2}\right)^{cc'}(\Gamma_J)_{\beta\alpha}\bar{q}\Gamma_J G^{A}_{\lambda\theta} q,$$

$$\Gamma^J = \mathbf{1},\ \gamma_5,\ \gamma_\mu,\ i\gamma_5\gamma_\mu,\ \sigma_{\mu\nu}/\sqrt{2}.$$

The LO part of the CF:

$$
\begin{aligned}
\Pi^{\mathrm{QCD}(1)}_{(\mathcal{S})\rho\sigma}(p,q) &= \frac{i}{4}\int d^4x e^{iq.x}\langle\mathcal{P}(q)|\bar{q}(0)\Gamma^J q(x)|0\rangle\Bigg\{\Bigg[\mathrm{Tr}\Big[\Gamma_J\tilde{S}^{(\mathrm{pert.})}_Q(x)\Big]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma}\\
&+ \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J S^{(\mathrm{pert.})}_Q(x)\gamma_5\Big)_{\rho\sigma} + t\Big\{\mathrm{Tr}\Big[\Gamma_J\gamma_5\tilde{S}^{(\mathrm{pert.})}_Q(x)\Big]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma}\\
&+ \mathrm{Tr}\Big[\Gamma_J\tilde{S}^{(\mathrm{pert.})}_Q(x)\gamma_5\Big]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} + \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J S^{(\mathrm{pert.})}_Q(x)\Big)_{\rho\sigma}\\
&- \Big(S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J\gamma_5 S^{(\mathrm{pert.})}_Q(x)\gamma_5\Big)_{\rho\sigma}\Big\} + t^2\Big\{\mathrm{Tr}\Big[\Gamma_J\gamma_5\tilde{S}^{(\mathrm{pert.})}_Q(x)\gamma_5\Big]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma}\\
&- \Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J\gamma_5 S^{(\mathrm{pert.})}_Q(x)\Big)_{\rho\sigma}\Big\}\Bigg] + \Big(Q\leftrightarrow Q'\Big)\Bigg\}, \qquad (17)
\end{aligned}
$$

$$\tilde{\Gamma}_J = C\Gamma_J^{\mathrm{T}}C$$

The one gluon exchange part form Q:

$$
\begin{aligned}
\Pi^{\mathrm{QCD(2a)}}_{(\mathcal{S})\rho\sigma}(p,q) &= -\frac{g_s}{12}\int d^4x \int \frac{d^4k}{(2\pi)^4}\int_0^1 du e^{i(q-k).x}\langle \mathcal{P}(q)|\bar{q}(x)\Gamma^J G_{\mu\nu}(ux)q(0)|0\rangle \\
&\times \Bigg\{\Bigg[\mathrm{Tr}\big[\Gamma_J\tilde{\Delta}^{\mu\nu}_Q(x)\big]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} + \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J\Delta^{\mu\nu}_Q(x)\gamma_5\Big)_{\rho\sigma} \\
&+ t\Big\{\mathrm{Tr}\big[\Gamma_J\gamma_5\tilde{\Delta}^{\mu\nu}_Q(x)\big]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma} + \mathrm{Tr}\big[\Gamma_J\tilde{\Delta}^{\mu\nu}_Q(x)\gamma_5\big]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} \\
&+ \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J\Delta^{\mu\nu}_Q(x)\Big)_{\rho\sigma} - \Big(S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J\gamma_5\Delta^{\mu\nu}_Q(x)\gamma_5\Big)_{\rho\sigma}\Big\} \\
&+ t^2\Big\{\mathrm{Tr}\big[\Gamma_J\gamma_5\tilde{\Delta}^{\mu\nu}_Q(x)\gamma_5\big]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma} - \Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J\gamma_5\tilde{\Delta}^{\mu\nu}_Q(x)\Big)_{\rho\sigma}\Big\}\Bigg] \\
&+ \Big(\Delta^{\mu\nu}_Q(x) \leftrightarrow S^{(\mathrm{pert.})}_{Q'}(x)\Big)\Bigg\}. \qquad (21)
\end{aligned}
$$

$$\tilde{\Delta}^{\mu\nu}_Q(x) = C\Delta^{T,\mu\nu}_Q(x)C.$$

One gluon exchange from Q'

- In the previous result just: Q←→Q'

- This leads to the appearance two and three particle nonlocal matrix elements:

$$\langle \mathcal{P}(q)|\bar{q}\Gamma_i q|0\rangle \quad \text{and} \quad \langle \mathcal{P}(q)|\bar{q}\Gamma_i G^A_{\lambda\tau} q|0\rangle,$$

- They are non-perturbative contributions of the CF

- Can be expressed in terms of DA's of different twists
- See

P. Ball and R. Zwicky, "New results on $B \to \pi, K, \eta$ decay formfactors from light-cone sum rules," Phys. Rev. D **71**, 014015 (2005).

P. Ball, "Theoretical update of pseudoscalar meson distribution amplitudes of higher twist: The Nonsinglet case," JHEP **9901**, 010 (1999).

$$
\begin{aligned}
\langle \pi(p)|\bar{q}(x)\gamma_\mu\gamma_5 q(0)|0\rangle &= -if_\pi p_\mu \int_0^1 du e^{i\bar{u}px}\left(\varphi_\pi(u) + \frac{1}{16}m_\pi^2 x^2 \mathbb{A}(u)\right) \\
&- \frac{i}{2} f_\pi m_\pi^2 \frac{x_\mu}{px}\int_0^1 du e^{i\bar{u}px}\mathbb{B}(u), \\
\langle \pi(p)|\bar{q}(x) i\gamma_5 q(0)|0\rangle &= \mu_\pi \int_0^1 du e^{i\bar{u}px}\varphi_P(u), \\
\langle \pi(p)|\bar{q}(x)\sigma_{\alpha\beta}\gamma_5 q(0)|0\rangle &= \frac{i}{6}\mu_\pi\left(1-\tilde{\mu}_\pi^2\right)\left(p_\alpha x_\beta - p_\beta x_\alpha\right)\int_0^1 du e^{i\bar{u}px}\varphi_\sigma(u),
\end{aligned}
$$

$$
\begin{aligned}
\langle\pi(p)|\bar{q}(x)\sigma_{\mu\nu}\gamma_5 g_s G_{\alpha\beta}(vx)q(0)|0\rangle &= i\mu_\pi\Big[p_\alpha p_\mu\left(g_{\nu\beta}-\frac{1}{px}(p_\nu x_\beta+p_\beta x_\nu)\right)\\
&- p_\alpha p_\nu\left(g_{\mu\beta}-\frac{1}{px}(p_\mu x_\beta+p_\beta x_\mu)\right)\\
&- p_\beta p_\mu\left(g_{\nu\alpha}-\frac{1}{px}(p_\nu x_\alpha+p_\alpha x_\nu)\right)\\
&+ p_\beta p_\nu\left(g_{\mu\alpha}-\frac{1}{px}(p_\mu x_\alpha+p_\alpha x_\mu)\right)\Big]\\
&\times \int \mathcal{D}\alpha e^{i(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{T}(\alpha_i),\\
\langle\pi(p)|\bar{q}(x)\gamma_\mu\gamma_5 g_s G_{\alpha\beta}(vx)q(0)|0\rangle &= p_\mu(p_\alpha x_\beta-p_\beta x_\alpha)\frac{1}{px}f_\pi m_\pi^2\int \mathcal{D}\alpha e^{i(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{A}_\parallel(\alpha_i)\\
&+ \Big[p_\beta\left(g_{\mu\alpha}-\frac{1}{px}(p_\mu x_\alpha+p_\alpha x_\mu)\right)\\
&- p_\alpha\left(g_{\mu\beta}-\frac{1}{px}(p_\mu x_\beta+p_\beta x_\mu)\right)\Big]f_\pi m_\pi^2\\
&\times \int \mathcal{D}\alpha e^{i(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{A}_\perp(\alpha_i),
\end{aligned}
$$

The generic configuration:

$$
\begin{aligned}
T_{[\quad,\alpha,\alpha\beta]}(p,q) &= i\int d^4x\int_0^1 dv\int \mathcal{D}\alpha e^{ip.x}(x^2)^n[e^{i(\alpha_{\bar{q}}+v\alpha_g)q.x}\mathcal{G}(\alpha_i), e^{iq.x}f(u)] \\
&\times \quad [1, x_\alpha, x_\alpha x_\beta]K_\mu(m_Q\sqrt{-x^2})K_\nu(m_Q\sqrt{-x^2}).
\end{aligned}
$$

To perform the Fourier transformation:

$$
\begin{aligned}
(x^2)^n &= (-1)^n \frac{d^n}{d\beta^n}\left(e^{-\beta x^2}\right)|_{\beta=0}, \\
x_\alpha e^{iP.x} &= (-i)\frac{d}{dP^\alpha} e^{iP.x}.
\end{aligned}
$$

For example

- One specific configuration is

$$\mathcal{Z}_{\alpha\beta}(p,q) = i\int d^4x \int_0^1 dv \int \mathcal{D}\alpha e^{i[p+(\alpha_{\bar{q}}+v\alpha_g)q].x}\mathcal{G}(\alpha_i)\left(x^2\right)^n \times x_\alpha x_\beta K_\mu(m_Q\sqrt{-x^2})K_\nu(m_Q\sqrt{-x^2}).$$

and the after Fourier and using the Borel transformation rule:

$$\mathcal{B}_{p1}(M_1^2)\mathcal{B}_{p2}(M_2^2)e^{b(p+uq)^2} = M^2\delta(b+\frac{1}{M^2})\delta(u_0-u)e^{\frac{-q^2}{M_1^2+M_2^2}},$$

$$u_0 = M_1^2/(M_1^2+M_2^2)$$

We have:

$$\mathcal{Z}_{\alpha\beta}(M^2) = \frac{i\pi^2 2^{4-\mu-\nu} e^{\frac{-q^2}{M_1^2+M_2^2}}}{M^2 m_{Q_1}^{2\mu} m_{Q_2}^{2\nu}} \int \mathcal{D}\alpha \int_0^1 dv \int_0^1 dz \frac{\partial^n}{\partial\beta^n} e^{-\frac{m_1^2\bar{z}+m_2^2 z}{z\bar{z}(M^2-4\beta)}} z^{\mu-1}\bar{z}^{\nu-1}(M^2-4\beta)^{\mu+\nu-1}$$
$$\times \delta[u_0-(\alpha_q+v\alpha_g)]\Big[p_\alpha p_\beta+(v\alpha_g+\alpha_q)(p_\alpha q_\beta+q_\alpha p_\beta)+(v\alpha_g+\alpha_q)^2 q_\alpha q_\beta$$
$$+\frac{M^2}{2}g_{\alpha\beta}\Big]. \qquad (28)$$

Subtraction:

$$e^{-\frac{m_1^2\bar{z}+m_2^2 z}{M^2 z\bar{z}}} = e^{-s_0/M^2}$$

s_0 is the continuum threshold

Therefore :

$$\int_0^1 dz \to \int_{z_{\rm min}}^{z_{\rm max}} dz$$

where

$$z_{\rm max(min)} = \frac{1}{2s_0}\left[(s_0+m_1^2-m_2^2)+(-)\sqrt{(s0+m_1^2-m_2^2)^2-4m_1^2 s_0}\right]$$

Putting all things together:

$$g_{B_1 B_2 \mathcal{P}}(M^2, s_0, t) = \frac{1}{\lambda_{B_1} \lambda_{B_2}} e^{\frac{m_{B_1}^2 + m_{B_2}^2}{2M^2}} \Pi_{B_1 B_2 \mathcal{P}}(M^2, s_0, t).$$

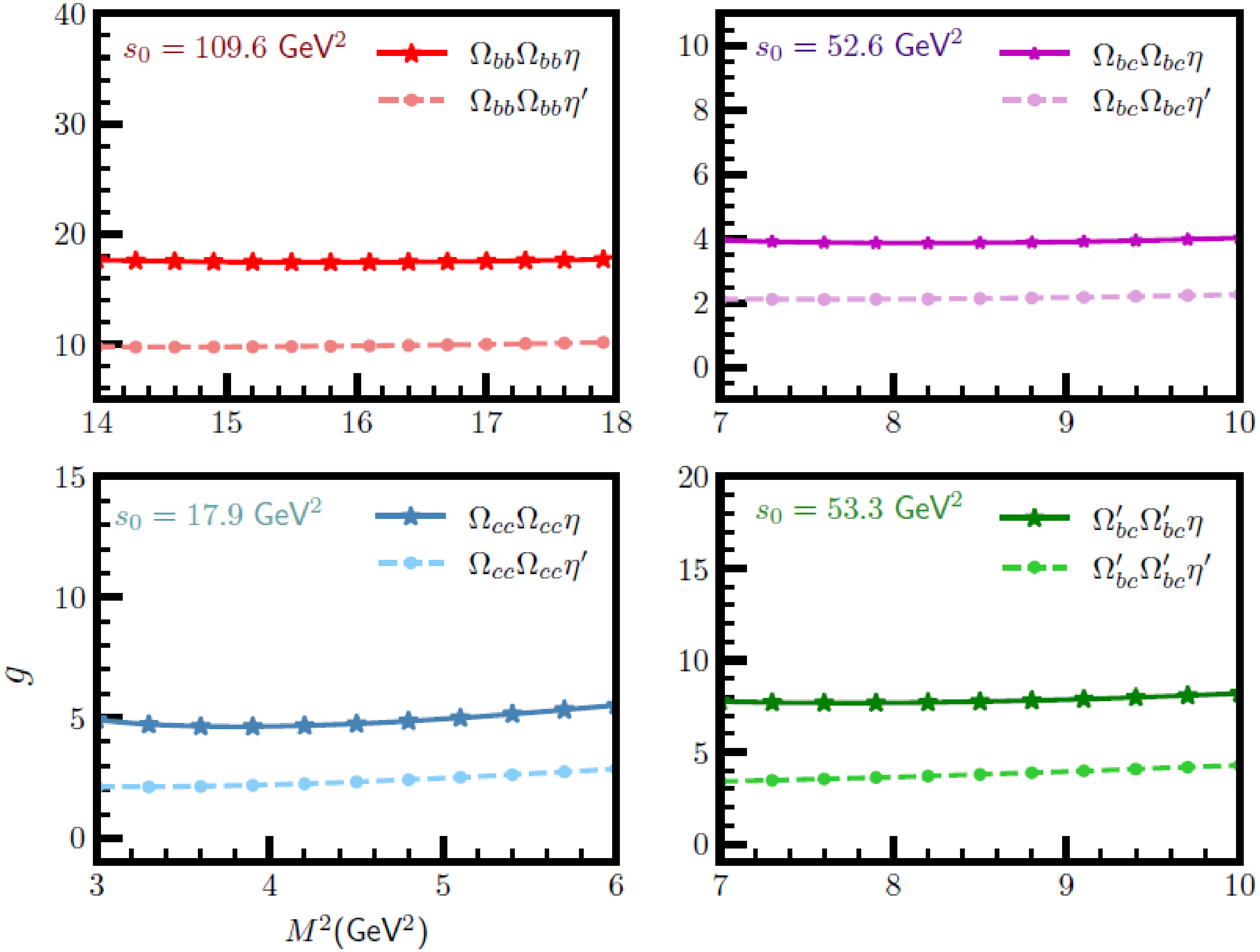
Three auxiliary parameters: M^2, s_0, t

For M^2

- The lower bound : the convergence of OPE

- The higher bound : pole dominance

$$R = \frac{\int_{(m_Q+m_{Q'})^2}^{s_0} ds \rho(s) e^{-s/M^2}}{\int_{(m_Q+m_{Q'})^2}^{\infty} ds \rho(s) e^{-s/M^2}} \geq \frac{1}{2}.$$



$s_0 = 109.6$ GeV^2
$\Omega_{bb}\Omega_{bb}\eta$
$\Omega_{bb}\Omega_{bb}\eta'$
$s_0 = 52.6$ GeV^2
$\Omega_{bc}\Omega_{bc}\eta$
$\Omega_{bc}\Omega_{bc}\eta'$
$s_0 = 17.9$ GeV^2
$\Omega_{cc}\Omega_{cc}\eta$
$\Omega_{cc}\Omega_{cc}\eta'$
$s_0 = 53.3$ GeV^2
$\Omega'_{bc}\Omega'_{bc}\eta$
$\Omega'_{bc}\Omega'_{bc}\eta'$
g
$M^2(\mathrm{GeV}^2)$

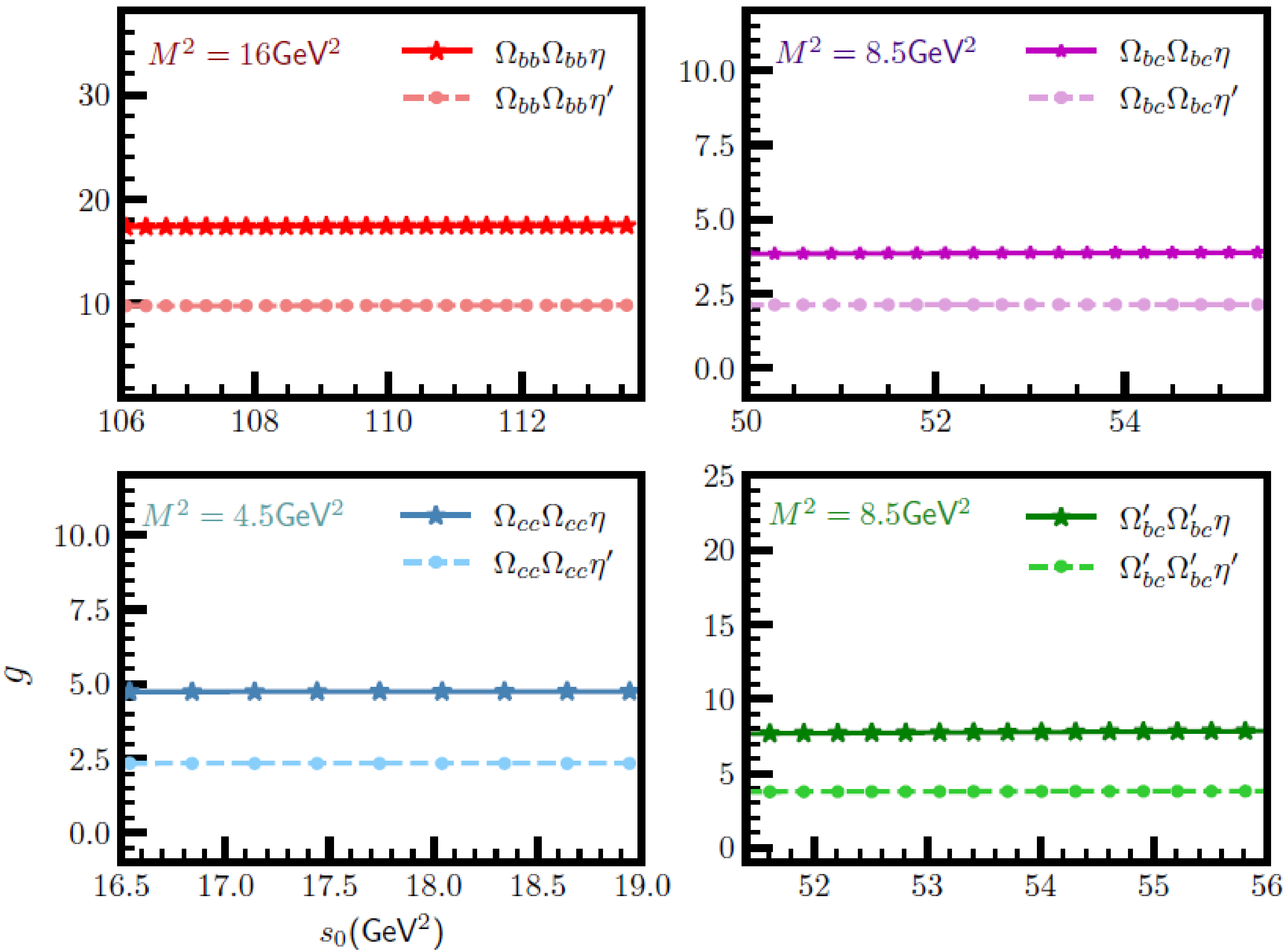
For s_0

- Including ground state and excluding excited states and continuum

- No experimental evidence for the first excited state

- We choose the interval in which the dependence of the results are weak:

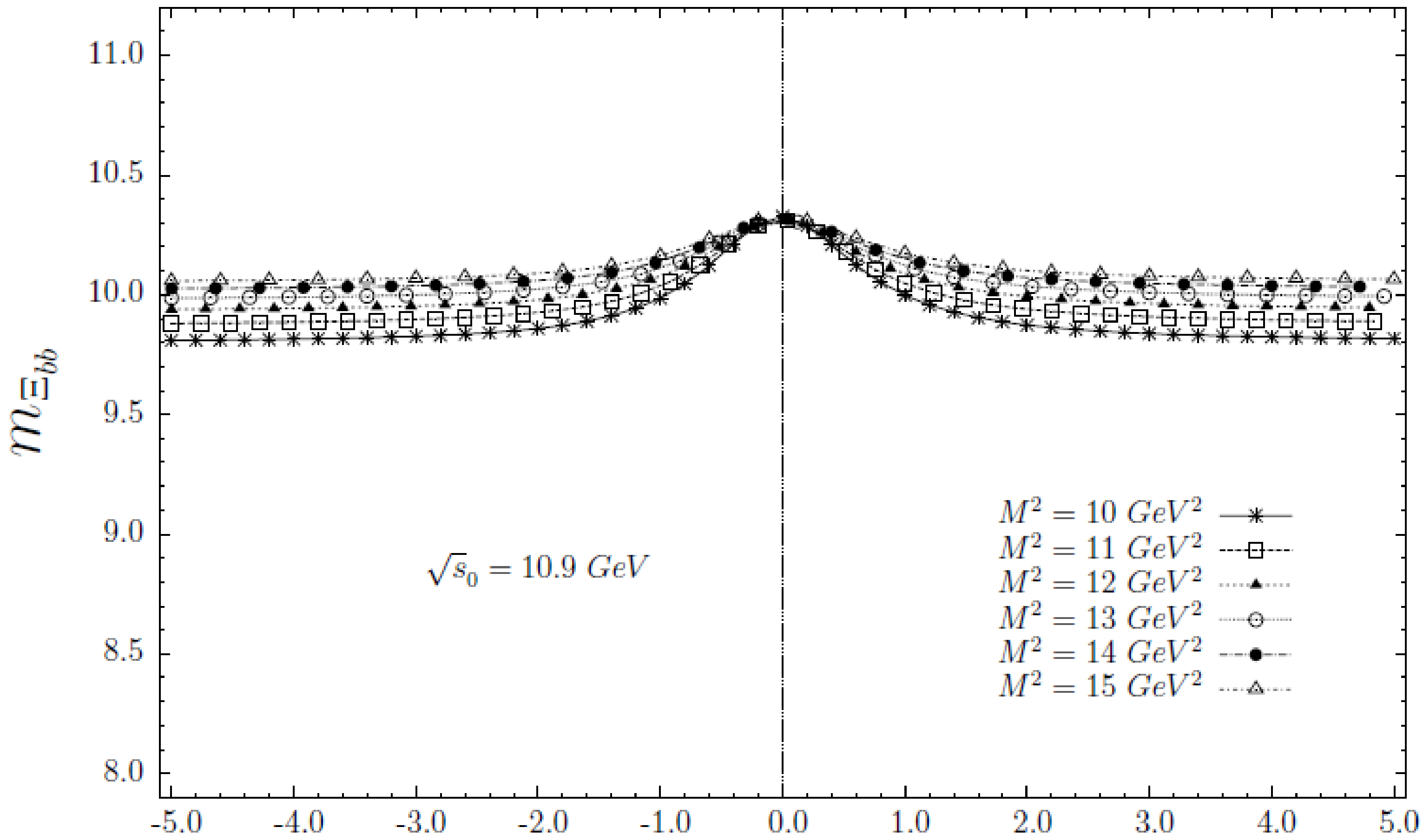
$$(m_B+0.3)^2 \leq s_0 \leq (m_B+0.7)^2 \mathrm{GeV}^2$$



$M^2 = 16\text{GeV}^2$
$\Omega_{bb}\Omega_{bb}\eta$
$\Omega_{bb}\Omega_{bb}\eta'$
$M^2 = 8.5\text{GeV}^2$
$\Omega_{bc}\Omega_{bc}\eta$
$\Omega_{bc}\Omega_{bc}\eta'$
$M^2 = 4.5\text{GeV}^2$
$\Omega_{cc}\Omega_{cc}\eta$
$\Omega_{cc}\Omega_{cc}\eta'$
$M^2 = 8.5\text{GeV}^2$
$\Omega'_{bc}\Omega'_{bc}\eta$
$\Omega'_{bc}\Omega'_{bc}\eta'$
g
$s_0(\text{GeV}^2)$

For t:

- The least dependence of mass and residues on t:



Channel	$M^2(\mathrm{GeV}^2)$	s_0 (GeV^2)	strong coupling constant
Decays to π			
$\Xi_{bb} \to \Xi_{bb}\pi^0$	$14 \leq M^2 \leq 18$	$105.3 \leq s_0 \leq 113.6$	$17.63^{0.38}_{0.24}$
$\Xi_{bb} \to \Xi_{bb}\pi^\pm$	$14 \leq M^2 \leq 18$	$105.3 \leq s_0 \leq 113.6$	$24.93^{0.53}_{0.33}$
$\Xi_{bc} \to \Xi_{bc}\pi^0$	$7 \leq M^2 \leq 10$	$49.3 \leq s_0 \leq 55$	$3.76^{0.17}_{0.10}$
$\Xi_{bc} \to \Xi_{bc}\pi^\pm$	$7 \leq M^2 \leq 10$	$49.3 \leq s_0 \leq 55$	$5.32^{0.24}_{0.14}$
$\Xi_{cc} \to \Xi_{cc}\pi^0$	$3 \leq M^2 \leq 6$	$15.4 \leq s_0 \leq 18.7$	$5.27^{0.97}_{0.70}$
$\Xi_{cc} \to \Xi_{cc}\pi^\pm$	$3 \leq M^2 \leq 6$	$15.4 \leq s_0 \leq 18.7$	$7.45^{1.37}_{0.98}$
$\Xi'_{bc} \to \Xi'_{bc}\pi^0$	$7 \leq M^2 \leq 10$	$50.3 \leq s_0 \leq 56.1$	$7.84^{0.24}_{0.24}$
$\Xi'_{bc} \to \Xi'_{bc}\pi^\pm$	$7 \leq M^2 \leq 10$	$50.3 \leq s_0 \leq 56.1$	$11.08^{0.33}_{0.34}$
$\Xi'_{bc} \to \Xi_{bc}\pi^0$	$7 \leq M^2 \leq 10$	$50.3 \leq s_0 \leq 56.1$	$0.62^{0.14}_{0.13}$
$\Xi'_{bc} \to \Xi_{bc}\pi^\pm$	$7 \leq M^2 \leq 10$	$50.3 \leq s_0 \leq 56.1$	$0.89^{0.20}_{0.19}$
Decays to K			
$\Omega_{bb} \to \Xi_{bb}\bar{K}^0$	$14 \leq M^2 \leq 18$	$105.3 \leq s_0 \leq 113.6$	$22.36^{1.30}_{0.91}$
$\Omega_{bb} \to \Xi_{bb}K^-$	$14 \leq M^2 \leq 18$	$105.5 \leq s_0 \leq 113.8$	$22.90^{1.31}_{0.91}$
$\Omega_{bc} \to \Xi_{bc}\bar{K}^0$	$7 \leq M^2 \leq 10$	$49.7 \leq s_0 \leq 55.5$	$4.04^{0.42}_{0.25}$
$\Omega_{bc} \to \Xi_{bc}K^-$	$7 \leq M^2 \leq 10$	$49.7 \leq s_0 \leq 55.5$	$4.05^{0.42}_{0.25}$
$\Omega_{cc} \to \Xi_{cc}\bar{K}^0$	$3 \leq M^2 \leq 6$	$16.2 \leq s_0 \leq 19.6$	$5.76^{1.40}_{0.80}$
$\Omega_{cc} \to \Xi_{cc}K^-$	$3 \leq M^2 \leq 6$	$16.2 \leq s_0 \leq 19.6$	$5.78^{1.42}_{0.84}$
$\Omega'_{bc} \to \Xi'_{bc}\bar{K}^0$	$7 \leq M^2 \leq 10$	$50.4 \leq s_0 \leq 56.2$	$11.11^{1.20}_{0.76}$
$\Omega'_{bc} \to \Xi'_{bc}K^-$	$7 \leq M^2 \leq 10$	$50.4 \leq s_0 \leq 56.2$	$11.14^{1.19}_{0.75}$

Decays to η			
$\Omega_{bb} \to \Omega_{bb}\eta$	$14 \leq M^2 \leq 18$	$105.3 \leq s_0 \leq 113.6$	$17.20\,^{0.75}_{0.75}$
$\Omega_{bc} \to \Omega_{bc}\eta$	$7 \leq M^2 \leq 10$	$49.7 \leq s_0 \leq 55.5$	$3.36\,^{0.25}_{0.15}$
$\Omega_{cc} \to \Omega_{cc}\eta$	$3 \leq M^2 \leq 6$	$16.2 \leq s_0 \leq 19.6$	$4.14\,^{0.90}_{0.48}$
$\Omega'_{bc} \to \Omega'_{bc}\eta$	$7 \leq M^2 \leq 10$	$50.4 \leq s_0 \leq 56.2$	$8.38\,^{0.64}_{0.42}$
Decays to η'			
$\Omega_{bb} \to \Omega_{bb}\eta'$	$14 \leq M^2 \leq 18$	$105.3 \leq s_0 \leq 113.6$	$9.54\,^{0.90}_{0.77}$
$\Omega_{bc} \to \Omega_{bc}\eta'$	$7 \leq M^2 \leq 10$	$49.7 \leq s_0 \leq 55.5$	$1.78\,^{0.24}_{0.18}$
$\Omega_{cc} \to \Omega_{cc}\eta'$	$3 \leq M^2 \leq 6$	$16.2 \leq s_0 \leq 19.6$	$1.80\,^{0.80}_{0.80}$
$\Omega'_{bc} \to \Omega'_{bc}\eta'$	$7 \leq M^2 \leq 10$	$50.4 \leq s_0 \leq 56.2$	$4.43\,^{0.76}_{0.62}$

Outlook

- Now we are able to analyze the bound-states with more tan one heavy quarks especially with the same heavy one.
- For example the strong coupling constants of Ξ_{cc} and Ξ_{bb} with π, ρ, K,… and their decay rates.
- Also the coupling of the heavy exotic states with standard mesons and baryons .

Thank You