# Hydrodynamization in the first-order causal and stable hydrodynamics

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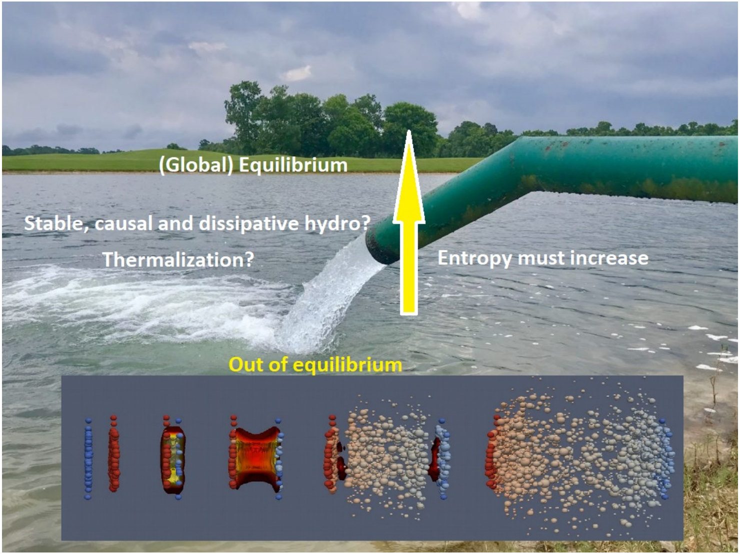
# Motivation

# What do we anticipate from hydrodynamics?



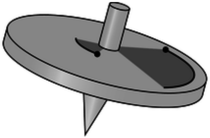

https://landassociation.org/texas-water-law-a-pond-to-call-my-own/, - MADAI collaboration, Hannah Petersen and Jonah Bernhard,

# Infinitely late-times: Global equilibrium?

"The Lie derivative of any physical observable along the four-temperature vector vanishes at thermodynamic equilibrium." (Becattini, 2016)
"A free-falling ideal thermometer in a fluid at global equilibrium will mark a constant temperature $1/(\beta.u)$ of the fluid ... with $u$ the four-velocity of the thermometer." (Becattini, 2016)

## Different ways to obtain

- Zubarev-Becattini: Maximize $S = -\mathrm{tr}(\rho \log \rho)$ with *some* constraints Becattini, 2016
- Poor man: Define $S^\mu$ from promoting $s$ and solve $\nabla \cdot S = 0$
- Or use kinetic theory De Groot et al., 1980
- Rigid motion: $\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \implies \beta_\mu = b_\mu + \omega_{\mu\nu} x^\nu$
- $b_\mu = u_\mu/T$ or $T = 1/\sqrt{\beta^2}$
- Hydrostatic configuration $u^\mu = (1, \mathbf{0})$ in some coordinate system, and $\beta_\mu = u_\mu/T$



$$E = -PV + TS + \mu N$$

$$\epsilon = -P + Ts + \mu n$$

# Conformal theories

In conformal theories, the four-temperature does not need to be a Killing vector. It is sufficient for it to be a conformal Killing vector for the system to be in global equilibrium. The physical observable need to have a definite conformal weight.

## Poor man's approach to conformal theories

- Energy-momentum tensor $T^{\mu\nu} \equiv \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta \log Z}{\delta g_{\mu\nu}}$
- Conformal symmetry $\implies g_{\mu\nu} T^{\mu\nu} = 0$
- Entropy current $S^\mu = P\beta^\mu - T^{\mu\nu}\beta_\nu$
- $\nabla \cdot S = 0 \implies \mathcal{L}_\beta g_{\mu\nu} = \frac{1}{2} \nabla \cdot \beta g_{\mu\nu}$
- Also $\mathcal{L}_\beta T = -\frac{1}{4} \nabla \cdot \beta T$
- (Almost) every physical quantity $\mathcal{L}_\beta X = -\frac{[X]}{4} \nabla \cdot \beta X$

## A general solution

- In global equilibrium it is guaranteed that $\nabla_\mu T^{\mu\nu} = 0$
- New definition $\mathcal{L}_\beta T + \frac{1}{4} \nabla \cdot \beta T = \frac{3}{4} \nabla \cdot u \left( f - \frac{2}{3} \right)$ with $f \equiv 1 + \frac{\dot{T}}{T \nabla \cdot u}$
- The general solution to ideal hydrodynamics is $f = 2/3$
- You may call this the **local equilibrium** if $\beta$ is not *still* a conformal Killing vector

# Your textbook hydro does not work!

"Landau-Lifshitz and Eckart first-order hydrodynamics (NS) predict that there is no equilibrium, and excitation can propagate faster than light." (For example: Hiscock et al., 1985)

"We may promote dissipative currents to dynamical variables." (Israel, 1975)

$$(\tau_\pi u^\alpha \partial_\alpha + 1)\, \pi^{\mu\nu} = -\eta \sigma^{\mu\nu} \qquad (\tau_\Pi u^\alpha \partial_\alpha + 1)\, \Pi = -\zeta \nabla \cdot u$$

"The relaxation time introduced by Müller-Israel-Stewart is not arbitrary." (Pu et al., 2009)

$$T\tau_\pi > 2\eta/s$$

"The first-order theory can be stable!" (Kovtun, 2019 and Bemfica et al., 2019)

"Unstable theories do not have an absolute maximum for the entropy." (Gavassino et al., 2020)

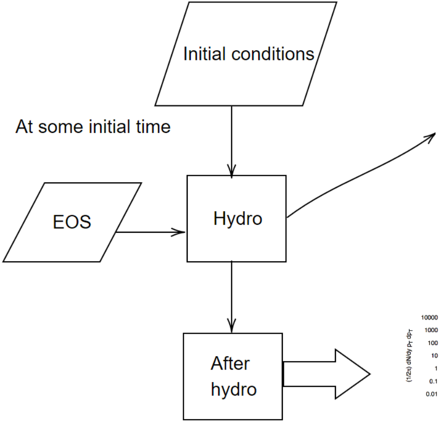
A lot of theories in the market

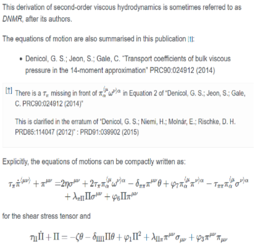
$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P})\, u^\mu u^\nu + \mathcal{P} g^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu + \pi^{\mu\nu} \quad \text{with} \quad \mathcal{E} = \epsilon + f_\mathcal{E} \quad \text{and} \quad \mathcal{P} = p + f_\mathcal{P}$$

| Prescription | Frame | $f_\mathcal{E}$ | $f_\mathcal{P}$ | $Q^\mu$ | $\pi^{\mu\nu}$ |
|---|---|---|---|---|---|
| Navier-Stokes | Landau | 0 | $-\zeta \nabla \cdot u$ | 0 | $-\eta\sigma^{\mu\nu}$ |
| MIS | Landau | 0 | $\Pi$ (dynamical variable) | 0 | dynamical variable |
| DNMR | Landau | 0 | $\Pi$ (dynamical variable) | 0 | dynamical variable |
| Disconzi | Landau | 0 | $-\zeta \nabla . C$ | 0 | $-2\eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\nabla_{(\alpha} C_{\beta)}$ $+\eta\Delta^{\mu\nu}\left(\frac{2}{3}\nabla \cdot C + \dot{F}\right)$ |
| FOCS | General | $\epsilon_1 \frac{\dot{T}}{T} + \epsilon_2 \nabla \cdot u$ | $\pi_1 \frac{\dot{T}}{T} + \pi_2 \nabla \cdot u$ | $\theta\left(a^\mu + \frac{1}{T}\nabla^\perp_\mu T\right)$ | $-\eta\sigma^{\mu\nu}$ |

# Hydro simulations made simple

Causal and stable dissipative hydrodynamics is required for heavy-ion collisions.



This derivation of second-order viscous hydrodynamics is sometimes referred to as *DNMR*, after its authors.

The equations of motion are also summarised in this publication [†]:

- Denicol, G. S.; Jeon, S.; Gale, C. "Transport coefficients of bulk viscous pressure in the 14-moment approximation" PRC90:024912 (2014)

[†] There is a $\tau_\pi$ missing in front of $\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}$ in Equation 2 of "Denicol, G. S.; Jeon, S.; Gale, C. PRC90:024912 (2014)"

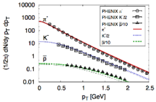
This is clarified in the erratum of "Denicol, G. S.; Niemi, H.; Molnár, E.; Rischke, D. H. PRD85:114047 (2012)" : PRD91:039902 (2015)

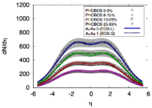
Explicitly, the equations of motions can be compactly written as:

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \varphi_7 \pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + \varphi_6 \Pi\pi^{\mu\nu}$$

for the shear stress tensor and

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \varphi_1\Pi^2 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} + \varphi_3\pi^{\mu\nu}\pi_{\mu\nu}$$



# Early thermalization puzzle

## Summary

- Be careful about the orders!
- $v_2$ sensitivity to $\tau_0 \to \tau_0 \leq 1\,\mathrm{fm/c}$ Romatschke, 2016
- NS hydro overestimates $S$, even with $\eta/s = 1/4\pi$ M.Lublinsky and E.Shuryak, 2007
- Proposal (that was not taken seriously): $scale \sim 1/T \to scale \sim 1/(TN_{\mathrm{dof}}^{1/3})$ M.Lublinsky and E.Shuryak, 2007
- Relatively fast thermalization for $Q_s \gg \Lambda_{QCD}$ (with $1+1$ expansion): Baier et al., 2001
- Their result: $\tau \sim 1.5\alpha_s^{-13}Q_s^{-1} \sim 6.9\,\mathrm{fm/c}$ !!!
- LRF $T^{\mu}_{\nu} = \mathrm{diag}\left(\mathcal{E}, P_T, P_T, P_L\right)$ $\qquad \mathcal{A} = \left(P_L - P_T\right)/P_{EQ}$
- Fast isotropization Peter Arnold et al., 2004
- Expansions is against isotropization
- In both weakly and strongly coupled regimes isotripization is not achieved in a short time
- But hydrodynamic works!

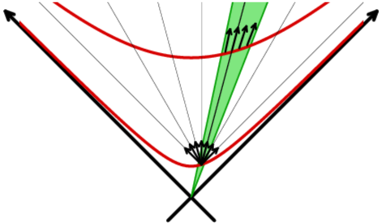
# Hydrodynamization

# (Conformal) Bjorken flow as a toy model

- $T^{\mu}_{\nu} = \mathrm{diag}\,(\mathcal{E}, P_T, P_T, P_L)$
- $\mathcal{A} = (P_L - P_T)\,/P_{EQ}$
- $T = \frac{1}{(\Lambda\tau)^{1/3}}\left(1 + \sum_{k=1}^{\infty} \frac{t_k}{(\Lambda\tau)^{2k/3}}\right)$
- Ideal hydro $T = \frac{1}{(\Lambda\tau)^{1/3}} \quad \mathcal{A} = 0$
- $C_\eta \equiv \eta/s \qquad C_\tau \equiv T\tau_\pi$
- MIS equation for Bjorken

$$\tau C_\tau \frac{\ddot{T}}{T} + 3\tau C_\tau \left(\frac{\dot{T}}{T}\right)^2 + \left(\frac{11 C_\tau}{3T} + \tau\right)\dot{T} - \frac{4C_\eta}{9\tau} + \frac{4C_\tau}{9\tau} + \frac{1}{3}T = 0.$$

- Exact solution for $C_\tau = 0$
- Divergent series for $C_\tau \neq 0$



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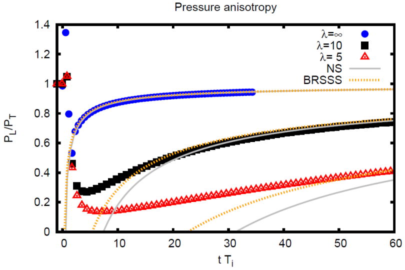
$$v_z = z/t = \tanh\xi \quad \tau = \sqrt{t^2 - z^2}$$
$$\epsilon(\tau) = 3P_{eq}(\tau) \sim T(\tau)^4$$
$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \tau^2\,\mathrm{d}\xi^2 + \mathrm{d}x^2 + \mathrm{d}y^2$$
$$u^\mu = (1, \mathbf{0})$$

# What is hydrodynamization?

"The timescale at which hydrodynamics first is able to closely approximate the subsequent dynamics of the exact underlying microscopic theory has been dubbed hydrodynamization time." (J. Casalderrey-Solana et al., 2011)

"Given the existence of a local rest frame, hydrodynamics offers a valid and quantitatively reliable description of the energy-momentum tensor even in non-equilibrium situations as long as the contribution from all non-hydrodynamic modes can be neglected." (P. Romatschke, 2017)

"The phenomenological success of hydrodynamics in describing experimental data from high energy nuclear collisions does not imply near-equilibrium behavior of the matter." (P. Romatschke, 2017)




Keegan et al., 2016

# Heller and Spalinski find attractors I

The nonlinear EOM in the MIS formalism forgets the initial condition Heller et al., 2015

- They were working on resurgence
- For simplicity they defined $w = T\tau$ and $f(w) = 1 + \tau\frac{\dot{T}}{T} = \tau\frac{\dot{w}}{w}$ Heller et al., 2011
- $\mathcal{A} = 18(f - 2/3)$
- EOM reduced to first-order:

$$C_\tau wff' + 4C_\tau f^2 + \left(w - \frac{16C_\tau}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_\tau}{9} - \frac{2w}{3} = 0$$

- The hydrodynamic gradient expansion

$$f(w) = \frac{2}{3} + \frac{4C_\eta}{9w} + \frac{8C_\eta C_\tau}{27w^2} + O(\frac{1}{w^3}).$$

First non-hydro mode:

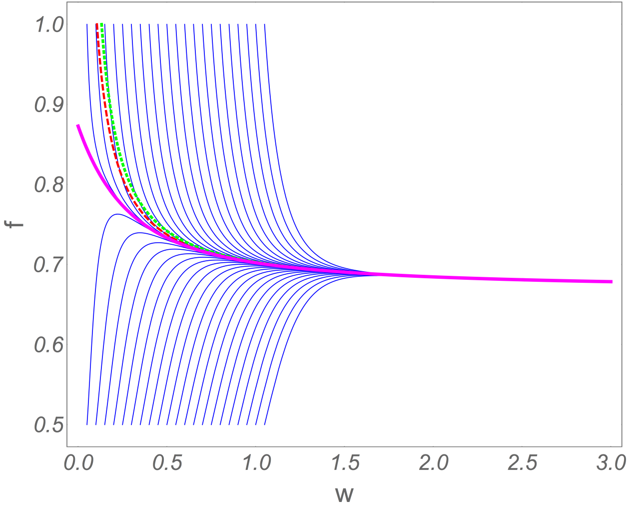
$$\delta f(w) \sim \exp\left(-\frac{3}{2C_\tau}w\right) w^{\frac{C_\eta - 2C_{\lambda_1}}{C_\tau}} \left(1 + O(\frac{1}{w})\right)$$

Initial condition:

$$f(w \ll 1) = \frac{2\sqrt{C_\tau} + \sqrt{C_\eta}}{3\sqrt{C_\tau}} + O(w)$$

# Heller and Spalinski find attractors II

Initial condition from $f(w \ll 1) = \frac{2\sqrt{C_\tau}+\sqrt{C_\eta}}{3\sqrt{C_\tau}} + O(w)$ vs arbitrary ones



# First order hydro is stable

# FOCS vs. Landau-Lifshitz

- $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \quad \dot{T} = u^\mu \partial_\mu T$
- $\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\nabla_\rho u_\sigma + \rho{\leftrightarrow}\sigma - (2/3) g_{\rho\sigma}\nabla\cdot u\right)$
- Conformal uncharged fluid $\epsilon = 3p = 3\underline{p}T^4 \quad \mu = 0 \quad \eta/s = C_\eta$
- Recall $T^{\mu\nu} = (\mathcal{E} + \mathcal{P})\, u^\mu u^\nu + \mathcal{P} g^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu + \pi^{\mu\nu}$
- Perturbing the hydrostatic: $\delta X \sim \delta\tilde{X}\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \overbrace{\Longrightarrow}^{EOM} M\delta X = 0$
- Dispersion relations: $\det(M) = \overbrace{F_{shear}}^{\delta\boldsymbol{v}\perp\boldsymbol{k}} \times \overbrace{F_{sound}}^{\delta\boldsymbol{v}\parallel\boldsymbol{k}} = 0 \implies \omega = \omega(\boldsymbol{k})$
- Causal: $\frac{1}{|\boldsymbol{k}|}\lim_{|\boldsymbol{k}|\to\infty}\mathrm{Re}(\omega) \le 1$ Stable: $\mathrm{Im}(\omega) < 0$ Nonhydro $\omega(\mathbf{0}) \neq 0$

## Landau-Lifshitz frame

- Fix the *frame*: $Q^\mu = 0$ and $\mathcal{E} = \epsilon$
- Go on-shell $3\dot{T} + T\nabla\cdot u = \mathcal{O}\left(\partial^2\right)$
- and $Ta^\mu + \Delta^{\mu\nu}\partial_\nu T = \mathcal{O}\left(\partial^2\right)$
- $\pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$

## FOCS

- Do not fix the frame and do not go on-shell
- Conformal invariance $f_\mathcal{E} = 3f_\mathcal{P}$
- $f_\mathcal{E} = 16C_\eta \boldsymbol{C_p}\underline{p}T^2(3\dot{T} + T\nabla\cdot u)$
- $Q^\mu = 16C_\eta \boldsymbol{C_Q}\underline{p}T^2(Ta^\mu + \Delta^{\mu\nu}\partial_\nu T)$
- $\pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$

# FOCS vs. Landau-Lifshitz

"The first-order hydrodyanmics can be stable by an expansion of parameter space." (Kovtun, 2019)

**Frame invariance:**[Gavassino et al., 2020]The hydro modes agree in all frames, but the nonhydro modes do not.

## Landau-Lifshitz frame

- Nonhydro mode in a moving frame: $\omega = \frac{iT\sqrt{1-v_0^2}}{v_0^2 C_\eta} + \mathcal{O}(\boldsymbol{k} \cdot \boldsymbol{v_0})$
- Instability in a moving frame
- Breaks causality in LRF
- Second law requires $C_\eta > 0$
- $\mathcal{A} = \frac{8C_\eta}{T\tau}$
- No absolute maximum for entropy

## FOCS

- Nonhydro mode in a moving frame: $\omega = \frac{iT\sqrt{1-v_0^2}}{(v_0^2 - 4C_Q)C_\eta} + \mathcal{O}(\boldsymbol{k} \cdot \boldsymbol{v_0})$
- Stable and Causal if $C_p > 1 \quad C_Q > 1 \quad C_\eta > 0$
- Off-shell parameters do not obey the second law
- $\mathcal{A} = \frac{8C_\eta}{T\tau}$
- Absolute maximum for entropy in equilibrium

## Where are we?

- Attractors emerge in different second-order hydrodynamic theories with conformal Bjorken and Guber flows Heller et al., 2017
- The same prescription (f) works for conformal FOCS theory Bemfica et al., 2017
- What does f mean in FOCS?
- Can FOCS work as an alternative to IS-like theories?

# Bjorken flow in FOCS hydrodynamics

# Power Series solution

- Bjorken symmetries $\implies Q^\mu = 0$
- EOM

$$4C_p C_\eta \ddot{T} + \dot{T}\left[T + 4C_p C_\eta\left(\frac{2\dot{T}}{T} + \frac{7}{3\tau}\right)\right] + \frac{T}{3\tau}\left(T + \frac{4(C_p-1)C_\eta}{3\tau}\right) = 0$$

- Final entropy distribution

$$\frac{dS}{dy} = \frac{4\underline{p}\pi R^2 T^3}{\tau_0^2 \Lambda^3} \times \left(\frac{1}{\Lambda\tau} + \frac{4C_p C_\eta \Lambda}{T}\left(1 - \left(\frac{1}{\Lambda\tau}\right)^{1/3}\frac{\dot{T}}{T}\right)\right)$$

- An exact (nonphysical) solution: $T = \frac{2C_\eta}{3\tau}(16C_p - 1)$
- **Truncated** series:

$$T_{(3)}(\tau) = \Lambda\left(\frac{1}{\Lambda\tau}\right)^{1/3}\left[1 - \frac{2C_\eta}{3}\left(\frac{1}{\Lambda\tau}\right)^{2/3} - \frac{16C_p C_\eta^2}{9}\left(\frac{1}{\Lambda\tau}\right)^{4/3} - \frac{256C_p^2 C_\eta^3}{27}\left(\frac{1}{\Lambda\tau}\right)^2\right]$$

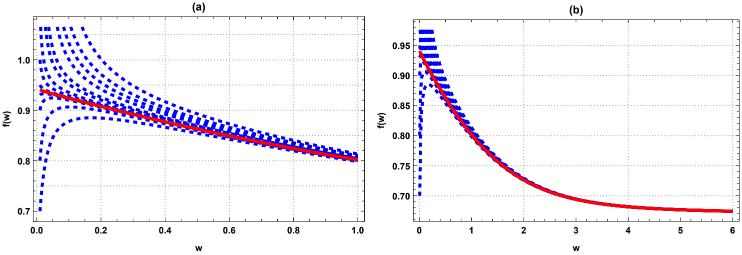
- Matching with $\mathcal{N} = 4$ SYM: $C_\eta = 1/4\pi \quad C_p = \frac{1-\log(2)}{4}$

# The attractor I

Function $f(w = T\tau)$ in FOCS theory is unrelated to the pressure anisotropy, instead it measures the decay of off-shell terms. M .S and F .Taghinavaz, 2020

- Recall $\mathcal{L}_\beta T + \frac{1}{4}\nabla\cdot\beta T = \frac{3}{4}\nabla\cdot u\left(f - \frac{2}{3}\right)$ with $f \equiv 1 + \frac{\dot{T}}{T\nabla\cdot u}$
- Related to damping of off-shell parts: $f_{\mathcal{P}} = \left(16\underline{p}C_\eta C_p\right)\left(T^3\nabla\cdot u\right)\left[f - \frac{2}{3}\right]$
- EOM: $4C_pC_\eta wff' + 12C_pC_\eta f^2 - \frac{56}{3}C_pC_\eta f + wf + \frac{64C_pC_\eta}{9} - \frac{4C_\eta}{9} - \frac{2w}{3} = 0$
- **Attractor**'s initial condition: $f(w) = \frac{7C_p + \sqrt{C_p(3+C_p)}}{9C_p} + \mathcal{O}(w)$
- No reheating: $C_p > 1 \implies f(0) > \frac{8}{9}$
- No early-time negative pressure: $\frac{P_L}{p} = 1 + \frac{16C_\eta}{w}\left[C_p\left(f(w) - 2\right) - 1\right]$
- Similar analytic structure of the Borel-transformed series with MIS
- $f(w) = \frac{2}{3} + \frac{4C_\eta}{9w} + \epsilon\delta f(w) \implies \delta f \sim \exp\left(-\frac{3w}{8C_pC_\eta}\right) w^{1+1/(4C_p)}$

# The attractor II



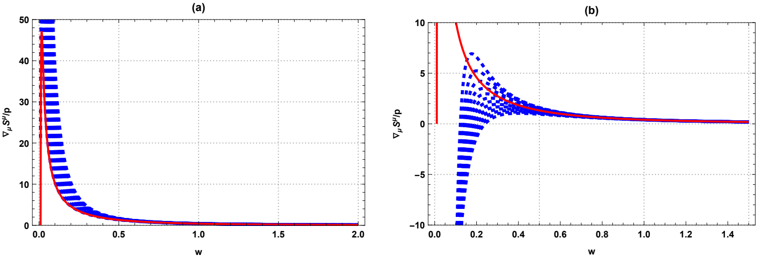

M. S and F. Taghinavaz, 2020

- In the MIS $RT > 1$ (Spalinksi, 2017) while in the FOCS $RT > 1.6$
- The slow-roll approximation:

$$f(w) = \frac{2}{3} - \frac{w}{24C_pC_\eta} + \frac{8C_pC_\eta + \sqrt{192C_pC_\eta^2 + (3w - 8C_pC_\eta)^2}}{27C_pC_\eta}$$

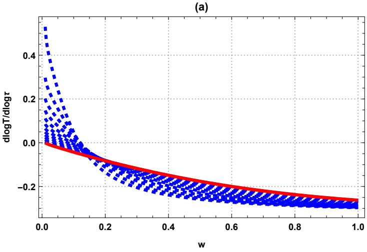
# The attractor III

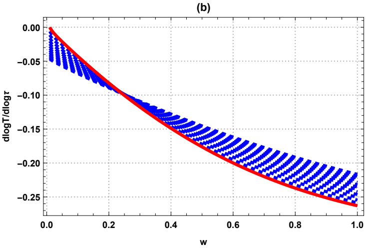
The off-shell transport parameters *need* to violate the second law of thermodynamics to preserve stability and causality.



# The attractor IV

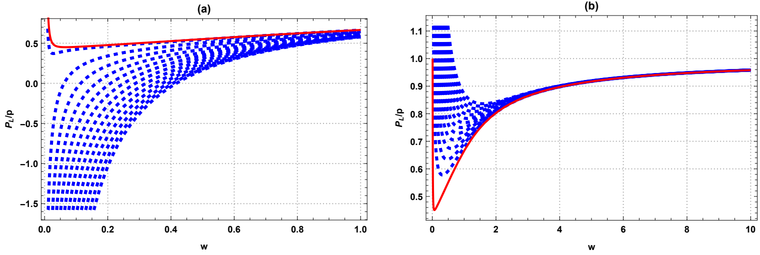
As a result, no reheating occurs for stable attractors at early-times.



# The attractor V

. . . and the pressure is always positive.



# Open questions

- What happens if conformal symmetry is broken?
- What happens in less symmetric flows?
- Attractors in MHD?
- Numerical simulations based on FOCS.