TRANSPORT COEFFICIENTS NEAR CRITICAL POINT

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HYDRODYNAMICS

- The effective **long distance and low frequency** description for a given classical or quantum many-body system at non-zero temperature.

$$\omega \to 0 \text{ as } k \to 0.$$

- Applies to the evolution which is slow, both in space and in time, relative to a certain microscopic scale. $k\ell_{\rm mfp}$
- the underlying microscopic theory → kinetic theory $\xRightarrow{Microscopic\ Scale}$
- $\ell_{\rm mfp}$: Mean free path
- k : Characteristic momentum scale

- the underlying microscopic theory might not admit a kinetic description $\xRightarrow{Microscopic\ Scale}$ $\ell_{\rm mfp} \sim T^{-1}$
- **T** : Temperature

HYDRODYNAMIC EQUATIONS

- local conservation laws

$$\partial_t \rho_a + \nabla \cdot \mathbf{J}_a = 0, \qquad (1)$$

- $\boldsymbol{\rho_a}$: Densities of locally conserved charges (Energy, Momentum, Particular Numbers)
- $\mathbf{J}_a$: Corresponding fluxes

CONSTITUTIVE RELATIONS

- **First view**

  Fluxes as a function of densities $\mathbf{J}_a = \mathbf{J}_a(\rho)$ (2)

- **Second view**

  Fluxes and densities as a function of conjugate quantities

- $\rho_a = \rho_a(\phi)$ $J_a(\phi) \equiv \mathrm{J}(\phi_a)$ (3)

- ϕ_a : Conjugate quantities to $\boldsymbol{\rho}_{\mathbf{a}}$ in the grand canonical ensemble(such as temperature, fluid velocity, and the chemical potential)

Having (2) **or** (3) $\rightrightarrows$ **Solving** (1)

CONSTITUTIVE RELATIONS

- **Zeroth order hydrodynamics(perfect fluid)**

Truncating the expansions at $O(\phi)$

- **First order hydrodynamics**

Truncating the expansions at $O(\nabla\phi)$

Second order hydrodynamics

Truncating the expansions at $O(\nabla^2\phi)$

- **Third order and so on**

$$T^{ab} = T^{ab}_{(0)}(u,T) + T^{ab}_{(1)}(\partial u, \partial T) + \ldots + T^{ab}_{(n)}(\partial^n u, \ldots, \partial^n T) + \ldots$$

Point:

going to higher orders in the derivative expansion improves the hydrodynamic description of the fluid

RELATIVISTIC HYDRODYNAMICS

- Hydrodynamic equations: express conservation laws of whatever is conserved
- Noether theorem: conservation laws are related to continuous symmetries of the fundamental microscopic theory, which imply the existence of conserved currents.
- The conserved current corresponding to the space-time translation symmetry is the energy momentum tensor $T^{\mu\nu}$
- A possible conserved current J^{μ} corresponding to a local U(1) symmetry
- The hydrodynamic equations :

$$\nabla_{\mu} T^{\mu\nu} = F^{\nu\mu} J_{\mu}, \qquad \nabla_{\mu} J^{\mu} = 0,$$

CONSTITUTIVE RELATIONS

- constitutive relations for the energy-momentum tensor and the current density (Any locally symmetric $T^{\mu\nu}$ and any locally J^{μ})

$$T^{\mu\nu} = \mathcal{E}u^{\mu}u^{\nu} + \mathcal{P}\Delta^{\mu\nu} + (q^{\mu}u^{\nu} + q^{\nu}u^{\mu}) + t^{\mu\nu} ,$$
$$J^{\mu} = \mathcal{N}u^{\mu} + j^{\mu} ,$$

$$\Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^{\mu}u^{\nu}$$

$$\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

- $\mathcal{E}$. $\mathcal{P}$. and $\mathcal{N}$: Scalar functions
- q^{μ} and j^{μ} : Transverse vectors $u_{\mu}q^{\mu} = u_{\mu}j^{\mu} = 0$.
- $t^{\mu\nu}$: Transverse, symmetric, traceless tensor
- The assumption of hydrodynamics enters in expressing the coefficients $\mathcal{E}, \mathcal{P}, q^{\mu}, t^{\mu\nu}, \mathcal{N},$ and j^{μ} in terms of the hydrodynamic variables $u_{\mu}, T,$ and μ

CONSTITUTIVE RELATIONS

- Constitutive relations for the energy-momentum tensor and the current density are written in the Landau frame

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \Pi^{\mu\nu},$$
$$J^\mu = n u^\mu + j^\mu,$$

- $\Pi^{\mu\nu}$ and j^μ : The dissipative contributions which can be expressed in terms of the derivatives of the hydrodynamic variables

CONSTITUTIVE RELATIONS

- If the underlying theory possess the conformal symmetry

$$\Pi^{\mu\nu} = -\eta\sigma^{\langle\mu\nu\rangle} + \eta\tau_\pi\left(D\sigma^{\langle\mu\nu\rangle} + \frac{1}{3}\sigma^{\langle\mu\nu\rangle}\nabla\cdot u\right) + \kappa\left(R^{\langle\mu\nu\rangle} - 2u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right)$$

$$+ \lambda_1^{(2)}\sigma^{\langle\mu}{}_\alpha\sigma^{\nu\rangle\alpha} + \lambda_2^{(2)}\sigma^{\langle\mu}{}_\alpha\Omega^{\nu\rangle\alpha} + \lambda_3^{(2)}\Omega^{\langle\mu}{}_\alpha\Omega^{\nu\rangle\alpha} + \sum_{i=1}^{20}\lambda_i^{(3)}\mathcal{O}_i^{(3)\mu\nu} + \mathcal{O}(\partial^4).$$

$$j^\mu = -\sigma T\Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \mathcal{O}(\partial^2).$$

$$\sigma^{\langle\mu\nu\rangle} \equiv \nabla^{\langle\mu}u^{\nu\rangle} \qquad A^{\langle\mu\nu\rangle} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}. \qquad \Omega^{\mu\nu} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha\right) \qquad D \equiv u^\mu\nabla_\mu$$

- η : Shear viscosity τ_π : Relaxation time σ : Conductivity
- $\lambda^{(2)}$: Non-linearized second order transport coefficients
- $\lambda_i^{(3)}$: Third order transport coefficients
- $\mathcal{O}_i^{(3)}$: Weyl-covariant tensors built out of the third order gradients

TRANSPORT COEFFICIENTS, VARIATIONAL APPROACH AND KUBO FORMULA

- Write down the constitutive relations and expand them up to the first order in the hydrodynamic fluctuations $(\delta T, \delta\mu, \delta u^{\mu})$ and $(\delta A_{\mu}, \delta g_{\mu\nu})$.
- Use the hydrodynamic equation to find the hydrodynamic fields in terms of source fields.
- Plug the solutions into the constitutive relations and make them on-shell .
- Use the on-shell currents to define the retarded Green's functions as

$$G^R_{J^\mu J^\nu} \equiv -\frac{\delta(\sqrt{-g}\langle J^\mu\rangle)}{\delta A_\nu}\bigg|_{\delta g=\delta A=0}, \qquad G^R_{J^\mu T^{\alpha\beta}} \equiv -2\frac{\delta(\sqrt{-g}\langle J^\mu\rangle)}{\delta h_{\alpha\beta}}\bigg|_{\delta g=\delta A=0},$$

$$G^R_{T^{\mu\nu} J^\alpha} \equiv -\frac{\delta(\sqrt{-g}\langle T^{\mu\nu}\rangle)}{\delta A_\alpha}\bigg|_{\delta g=\delta A=0}, \qquad G^R_{T^{\mu\nu} T^{\alpha\beta}} \equiv -2\frac{\delta(\sqrt{-g}\langle T^{\mu\nu}\rangle)}{\delta h_{\alpha\beta}}\bigg|_{\delta g=\delta A=0}.$$

- Each Green's function corresponds to a specific set of transport coefficients.

TRANSPORT COEFFICIENTS, VARIATIONAL APPROACH AND KUBO FORMULA

- A $3+1$ dimensional hydrodynamic theory with a $\mathrm{SO}(3)$ symmetry in the spacial directions.
- Using this symmetry we can set the momentum along the z coordinate.
- We will focus on the shear retarded Green's function $G^R_{xy,xy} \equiv G^R_{T^{xy}T^{xy}}$
- It has the following gradient expansion up to the third order

$$G^R_{xy,xy}(\omega, q) = p - i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}\left(\omega^2 + q^2\right) - \frac{i}{2}\lambda^{(3)}_{17}\omega^3 + \frac{i}{2}\left(\lambda^{(3)}_1 - \lambda^{(3)}_{16} - \lambda^{(3)}_{17}\right)\omega q^2.$$

KUBO FORMULA

- Kubo formula for each transport coefficient in this channel

$$\eta = -\lim_{\omega\to 0}\lim_{q\to 0}\frac{d}{d\omega} Im\, G^R_{xy,xy},$$

$$\kappa = -2\lim_{\omega\to 0}\lim_{q\to 0}\frac{d}{dq^2} Re\, G^R_{xy,xy},$$

$$\eta\tau_\Pi - \frac{\kappa}{2} = \lim_{\omega\to 0}\lim_{q\to 0}\frac{d}{d\omega^2} Re\, G^R_{xy,xy},$$

$$\lambda^{(3)}_{17} = -2\lim_{\omega\to 0}\lim_{q\to 0}\frac{d}{d\omega^3} Im\, G^R_{xy,xy},$$

$$\lambda^{(3)}_1 - \lambda^{(3)}_{16} - \lambda^{(3)}_{17} = 2\lim_{\omega\to 0}\lim_{q\to 0}\frac{d^2}{d\omega dq^2} Im\, G^R_{xy,xy}.$$

HOLOGRAPHIC MODEL: 1RCBH

- The 1RCBH model is described by the following Einstein-Maxwel-dilaton (EMD) action

$$S_{\rm bulk} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - \frac{f(\phi)}{4}F_{MN}F^{MN} - \frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi)\right]$$

- G_5 : The five dimensional Newton's constant
- The self-interacting dilaton potential V (φ) and the Maxwell-dilaton coupling f(φ) are given by

$$V(\phi) = -\frac{1}{L^2}\left(8e^{\frac{\phi}{\sqrt{6}}} + 4e^{-\sqrt{\frac{2}{3}}\phi}\right)$$

$$f(\phi) = e^{-2\sqrt{\frac{2}{3}}\phi}.$$

- L : The AdS_5 radius

HOLOGRAPHIC MODEL: IRCBH

- The charged static stationary black brane solutions are given by

$$ds^2 = e^{2A(\tilde{r})}\left(-h(\tilde{r})dt^2 + d\vec{x}^2\right) + \frac{e^{2B(\tilde{r})}}{\tilde{r}^4 h(\tilde{r})} d\tilde{r}^2,$$

$$A(\tilde{r}) = -\log\tilde{r} + \frac{1}{6}\log\left(1 + \widetilde{Q}^2\tilde{r}^2\right),$$

$$B(\tilde{r}) = \log\tilde{r} - \frac{1}{3}\log\left(1 + \widetilde{Q}^2\tilde{r}^2\right),$$

$$h(\tilde{r}) = 1 - \frac{\widetilde{M}^2\tilde{r}^4}{1 + \widetilde{Q}^2\tilde{r}^2},$$

$$\phi(\tilde{r}) = -\sqrt{\frac{2}{3}}\log\left(1 + \widetilde{Q}^2\tilde{r}^2\right),$$

$$\mathbf{A}(\tilde{r}) = \widetilde{M}\widetilde{Q}\left(\frac{\tilde{r}_h^2}{1 + \widetilde{Q}^2\tilde{r}_h^2} - \frac{\tilde{r}^2}{1 + \widetilde{Q}^2\tilde{r}^2}\right) dt,$$

$$M \equiv r_h^2\widetilde{M}, \qquad Q \equiv r_h\widetilde{Q}, \qquad r \equiv r_h\tilde{r}$$

$$\tilde{r}_h = \sqrt{\frac{\sqrt{4\widetilde{M}^2 + \widetilde{Q}^4} + \widetilde{Q}^2}{2\widetilde{M}^2}},$$

the horizon radius can be fixed to one

$$M = \sqrt{1 + Q^2}.$$

- $\tilde{r} = 0$: the boundary of asymptotically AdS_5

  $\widetilde{M}$: Mass of the black brane $\widetilde{Q}$: Charge of the black brane

HOLOGRAPHIC MODEL: 1RCBH

- The Hawking temperature T of the black brane (the e temperature of the boundary theory)

$$T = \frac{2+Q^2}{2\pi \tilde{r}_h \sqrt{1+Q^2}},$$

- The chemical potential of the dual theory

$$\mu = \lim_{r\to 0} \mathbf{A}_t(r) = \frac{Q}{\tilde{r}_h \sqrt{1+Q^2}}.$$

- One can characterize the 1RCBH model either by two non-negative parameters :

From gravity point of view (M,Q) , From the boundary point of view (μ, T)

PHASE DIAGRAM

- The class of solutions corresponding to the 1RCBH model may be parametrized by different values of $\frac{\mu}{T}$

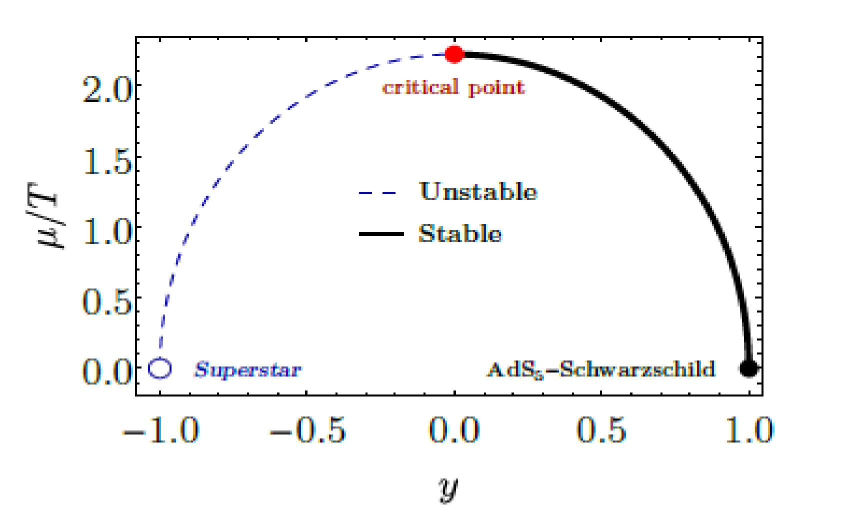
$$Q = \sqrt{2}\frac{1 \pm \sqrt{1 - \left(\frac{\mu/T}{\pi/\sqrt{2}}\right)^2}}{\left(\frac{\mu/T}{\pi/\sqrt{2}}\right)}.$$

- the solutions with −/+ sign in equation are thermodynamically stable/unstable black branes.
- Since Q is real and nonnegative → $\frac{\mu}{T} \in [0, \pi/\sqrt{2}]$ and for a given value of Q there are two different corresponding values of $\frac{\mu}{T} \in [0, \pi/\sqrt{2}]$ which parametrize two different branches of solutions.

The point of the phase diagram where these two branches merge $\mu/T = \pi/\sqrt{2}$ or $Q = \sqrt{2}$

PHASE DIAGRAM

- Introducing a new variable y as $$y^2 + \frac{2}{\pi^2}\left(\frac{\mu}{T}\right)^2 = 1, \qquad y \in [0, 1],$$



$y = 0$: parametrizes the critical background geometry

$y = 1$: parametrizes the AdS5-Schwarzschild background

THERMODYNAMIC OF IRCBH

- one may express the charge Q, the Hawking temperature T and the chemical potential μ of the black brane in terms of this new dimensionless parameter

$$Q=\sqrt{2\frac{1-y}{1+y}},\qquad T=\frac{2}{\pi\tilde{r}_h\sqrt{(3-y)(1+y)}},\qquad \mu=\frac{\pi T\sqrt{1-y^2}}{\sqrt{2}}.$$

- Bekenstein entropy density, the R-charge density and the pressure

$$\frac{s}{N_c^2T^3}=\frac{\pi^2}{16}(3-y)^2(1+y)\qquad \frac{\rho}{N_c^2T^3}=\frac{\sqrt{2}}{32}\sqrt{1-y^2}(3-y)^2.\qquad \frac{p}{N_c^2T^4}=\frac{\pi^2}{128}(3-y)^3(1+y)$$

- the Hessian matrix of the boundary thermodynamic quantities

$$\frac{\partial(s,\rho)}{\partial(T,\mu)}=\begin{pmatrix}\frac{\partial s}{\partial T} & \frac{\partial s}{\partial \mu}\\ \frac{\partial \rho}{\partial T} & \frac{\partial \rho}{\partial \mu}\end{pmatrix}\qquad \frac{\mathcal{J}}{N_c^4T^4}=\frac{3\pi^2}{256}(3-y)^4\left(1+\frac{1}{y}\right)$$

- the Jacobian is divergent at $y=0\ (\frac{\mu}{T}=\frac{\pi}{\sqrt{2}})$ignaling that there is a second order

phase transition.

TRANSPORT COEFFICIENT

- Spin 2 sector: Fluctuations which are tensors under $SO(3)$. The retarded Green's function

$$G^R_{xy,xy} = \frac{N_c^2 \pi^2 T^4}{4} \left[\frac{(3-y)^3(1+y)}{32} - \frac{2i\mathfrak{w}\alpha^4}{1+y} - \frac{4\alpha^4 \mathfrak{q}^2}{(1+y)(3-y)} + \frac{4\mathfrak{w}^2 \alpha^4 (1 - 2\log 2 + \log(1+y))}{(3-y)(1+y)} - \frac{i\mathfrak{w}^3}{8}\lambda^{(3)}_{17} + i\mathfrak{w}\mathfrak{q}^2 (\lambda^{(3)}_1 - \lambda^{(3)}_{16} - \lambda^{(3)}_{17}) \right].$$

- The transport coefficient

$$\eta = \frac{\pi N_c^2 T^3 (3-y)^2 (1+y)}{64},$$

$$\kappa = \frac{N_c^2 T^2 (3-y)(1+y)}{32},$$

$$\tau_\pi = \frac{2 - 2\log(2) + \log(1+y)}{\pi T (3-y)},$$

$$\lambda^{(3)}_{17} = \frac{N_c^2 T (1+y)}{128\pi} \left(\log\left(\frac{4}{1+y}\right)\left(8 + (1+y)\log\left(\frac{1+y}{4}\right)\right) + 2(3-y)\mathrm{Li}_2\left(\frac{3-y}{4}\right)\right),$$

$$\lambda^{(3)}_1 - \lambda^{(3)}_{16} = \frac{N_c^2 T (1+y)}{128\pi} \left(\log\left(\frac{4}{1+y}\right)\left(16 + (1+y)\log\left(\frac{1+y}{4}\right)\right) + 2(3-y)\mathrm{Li}_2\left(\frac{3-y}{4}\right)\right)$$

TRANSPORT COEFFICIENT

- the shear viscosity satisfies the universal relation

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$

- The $y = 1$ limit of those transport coefficients coincide with the counterparts of the $\mathcal{N} = 4$ SYM theory dual to AdS5 black brane background.

TRANSPORT COEFFICIENT

- Spin 1 sector: Fluctuations which are vectors under $\mathrm{SO}(3)$

$$\theta_1 \equiv -\left(\lambda_1^{(3)} + \lambda_2^{(3)} + \lambda_4^{(3)}\right) = \frac{N_c^2 T}{32\pi} y.$$

- At the critical point of the phase transition $y = 0 \; (\frac{\mu}{T} = \frac{\pi}{\sqrt{2}})$ the θ_1 vanishes.

  This might be a hint to the symmetry enhancement of the underlying theory at the critical point.

- One may compare this phenomenon with vanishing bulk viscosity for theories with conformal symmetry.

CRITICAL EXPONENT

- If a quantity $\mathcal{P}$ has the following expression near the critical point in terms of the dimensionless parameter $y = \sqrt{1 - \frac{2\mu^2}{\pi^2 T^2}}$

$$\mathcal{P} - \mathcal{P}_c \sim y^m$$

then, the behaviour of the associated quantity near critical point at either fixed temperature or fixed chemical potential will be given by

$$\mathcal{P} - \mathcal{P}_c \sim |\mu - \mu_c|^{m/2}, \qquad T = \text{fixed},$$
$$\mathcal{P} - \mathcal{P}_c \sim |T - T_c|^{m/2}, \qquad \mu = \text{fixed}.$$

- $\mathcal{P}_c$: the critical value of the quantity for $m < 2$ there is a critical exponent $\theta = \frac{m}{2}$

CRITICAL EXPONENT

- the behavior of transport coefficients near the critical point

$$\eta \simeq \frac{\pi N_c^2 T_c^3}{8}\left(\frac{9}{8}+\frac{3}{8}y\right), \qquad \kappa \simeq \frac{N_c^2 T_c^2}{8}\left(\frac{3}{4}+\frac{1}{2}y\right),$$

$$\tau_\pi \simeq \frac{2-\log(2)}{2\pi T_c}\left(\frac{4\log(2)-4}{3\log(2)-6}+\frac{4\log(2)-10}{9\log(2)-18}y\right), \qquad \theta_1 \simeq \frac{N_c^2 T_c}{32\pi}y,$$

$$\lambda_{17}^{(3)} \simeq \frac{N_c^2 T_c\left(\pi^2-12(\log(2)-2)\log(2)\right)}{192\pi}\times \left(\frac{9\mathrm{Li}_2\left(\frac{3}{4}\right)-6(\log(2)-4)\log(2)}{\pi^2-12(\log(2)-2)\log(2)}+\frac{3\left(2\mathrm{Li}_2\left(\frac{3}{4}\right)-4-4\log^2(2)+8\log(2)\right)}{\pi^2-12(\log(2)-2)\log(2)}y\right)$$

$$\lambda_1^{(3)}-\lambda_{16}^{(3)} \simeq \frac{N_c^2 T_c\left(\pi^2-12(\log(2)-4)\log(2)\right)}{192\pi}\times \left(\frac{9\mathrm{Li}_2\left(\frac{3}{4}\right)-6(\log(2)-8)\log(2)}{\pi^2-12(\log(2)-2)\log(2)}+\frac{3\left(2\mathrm{Li}_2\left(\frac{3}{4}\right)-4-2\log^2(2)+8\log(2)\right)}{\pi^2-12(\log(2)-2)\log(2)}y\right)$$

- All the quantities are linear in *y* which means that they have the same dynamical critical exponent $\theta = \frac{1}{2}$.

CONCLUSION

- To derive the transport coefficients for the dual theory of 1RCBH up to the third order of gradient expansion.

- To find that all the hydrodynamic quantities have the same critical exponent near the critical point $\theta = \frac{1}{2}$.

- To propose a relation between symmetry enhancement of the underlying theory and vanishing of the only third order hydrodynamic transport coefficient (θ_1 = 0).

THANK YOU