1 Credit

These notes are for teaching purposes, and draw shamelessly on available reference material. The main references are listed at the end of these notes.

2 Defining emittance

Emittance is the property of a particle beam that characterizes its size. Roughly, emittance is an area or volume in the phase space of the particles. There are two phase space variables for each spatial direction. The phase space variables for a particle are x, p_x, y, p_y, z, and p_z with time as the independent variable. These coordinates correspond to the position and momentum components of the particle. Often the coordinates are taken to be the errors in position and momentum with respect to an ideal particle. For example, an ideal particle would have no transverse momentum component, $p_x = 0$. Its position would lie along the ideal trajectory through the machine and be defined as $x = 0$. Longitudinally, the energy or momentum of a particle is defined to be the difference from the ideal (non-zero) momentum or energy.

Emittance is used to describe a beam because unlike the physical dimensions of the beam, which vary with location in an accelerator, emittance is invariant in the absence of dissipative or cooling forces. Sometimes motion in each plane (2 transverse, 1 longitudinal) is very weakly coupled; in this case it may be possible to treat the motion in each plane independently. As this case is the simplest, it will be used for examples and discussion here.

2.1 Particle distributions

The distribution function of a beam, $f_6(x, y, z, p_x, p_y, p_z; t)$, describes the coordinate distribution of particles in a beam. Integrating the distribution function over a region of phase space gives the number of particles found in that region of phase space [2]:

$$dN = f(\vec{q}, \vec{p}, t)d^3qd^3p$$

A commonly used distribution function to represent a particle beam is the Gaussian distribution. In transverse phase space of horizontal motion (for example), the functional form of a bi-Gaussian is given by the following:

$$f(x, p_x) = \frac{1}{2\pi\sigma_x\sigma_{p_x}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \exp\left(-\frac{p_x^2}{2\sigma_{p_x}^2}\right) \tag{1}$$

The peaks of the distribution hopefully correspond to those particles having no error ($x = 0, p_x = 0$). The distribution is symmetric, so there are as many particles with positive errors as with negative errors. The standard deviation, σ, for the case when the mean value of the distribution is zero ($< x >= 0$), is as follows:

$$\sigma \equiv \sqrt{\sigma^2} =< x^2 >^{\frac{1}{2}} = \lim_{N\to\infty} \left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$$

The summation is over all particles, and x_i is the position of the i^{th} particle.

The bi-gaussian integrated over all horizontal phase space is one, representing 100% of the particles. Integrating the distribution over some smaller phase space area yields a fraction that represents the fraction of the particles enclosed.

$$\frac{1}{2\pi\sigma_x\sigma_{p_x}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_x^2}} \ dx \int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2\sigma_{p_x}^2}} \ dp_x = \frac{1}{2\pi\sigma_x\sigma_{p_x}} \left[\sqrt{2\pi}\sigma_x\right] \left[\sqrt{2\pi}\sigma_{p_x}\right] = 1$$

2.2 Phase space

A phase space plot has a point plotted for each particle in the beam at the phase space coordinates corresponding to the particles. A phase space plot is for a specific location (time) in the machine, since the particle positions and momenta evolve as a beam propagates. If motion in each plane is independent of the others, then there is a horizontal phase space (x vs. p_x), and a 2D horizontal emittance can be extracted from the area of phase space covered by the beam. Similarly, there is a vertical and longitudinal emittance. A single experimental measurement of the beam dimension (number of particles versus position) cannot by itself yield phase space information, since only the particle positions, not the momenta, are known from a single measurement. Typically transverse experimental apparatus must be present in at least two locations, so that the needed angle (momentum) information can be obtained from the particle trajectories. High energy machines with well understood optics can be an exception to this rule. The positions of the particles can be measured, while the needed angle information is based on the machine optics constraining the particle motion.

2.3 High energy beams

When the transverse momentum components are small compared to the longitudinal, $p_z >> p_x, p_y$, and p_z can be considered to be the same for all particles, then the paraxial approximation can be applied to the beam motion. In this case, $x^{'}$, $y^{'}$ can be taken to be the transverse angles with respect to the ideal trajectory, $x^{'} = \tan(\alpha_x) \sim \alpha_x$ and similarly for $y^{'}$. Then, for the horizontal coordinates, p_x is related to $x^{'}$ in the following way:

$$p_x = mv_x + qA_x = m_0 c\beta\gamma x^{'} + qA_x$$

where m_0 is the rest mass of the particle, $x^{'} = \frac{dx}{dz}$, q is the charge of the particle, and A_x is the x-component of the magnetic vector potential. A similar relation holds for p_y. The magnetic vector potential is oriented perpendicularly to the associated magnetic field. If the magnetic fields through which the beam travels are entirely transverse (no solenoids), $\vec{B} = B_x\hat{i} + B_y\hat{j}$, and $A_x = A_y = 0$. Then, p_x may be written:

$$p_x = m_0 c\beta\gamma x^{'} \tag{2}$$

'Trace space' (a term used to make a distinction with phase space) is a coordinate space using $(x, x^{'}, y, y^{'})$ to describe the transverse motion instead of (x, p_x, y, p_y).

2.4 Transverse emittance from Hill's equation

The formalism used in this section to derive an expression for the transverse beam emittance (assuming uncoupled motion) is typically used for high energy synchrotrons.

Consider the case of uncoupled motion where in each transverse dimension the beam dynamics may be described by the 1D Hill's equation. The solution of the equation for the *single particle* transverse position (as a function of s, the longitudinal location) is the following:

$$x(s) = A\sqrt{\beta(s)}\cos(\psi(s)) \tag{3}$$

where A describes the amplitude dependence on initial conditions, and $\sqrt{\beta(s)}$ describes the amplitude dependence on the machine lattice. In other words, $\beta(s)\cos(\psi(s))$ varies around the machine, but is the same for every particle in the beam. On the other hand, A is the same everywhere in the machine, but every particle has a different amplitude of motion, A. The optical parameter, $\beta(s)$, is called a Twiss parameter. The other two Twiss parameters, $\alpha(s)$ and $\gamma(s)$, are functions of β and are defined below. An equation governing the phase space trajectory of a particle can be found using the solution (Eq.(3)) to construct a constant of the motion (do this in Exercise 4 of the transverse emittance tutorial). The result can be cast in the form of an equation for a circle in the $x - (\beta x' + \alpha x)$ plane, with radius $r = A\sqrt{\beta(s)}$:

$$(\beta x' + \alpha x)^2 + x^2 = A^2 \beta \tag{4}$$

Where x' is the angle coordinate of the particle, and α is defined by the following,

$$\alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds}$$

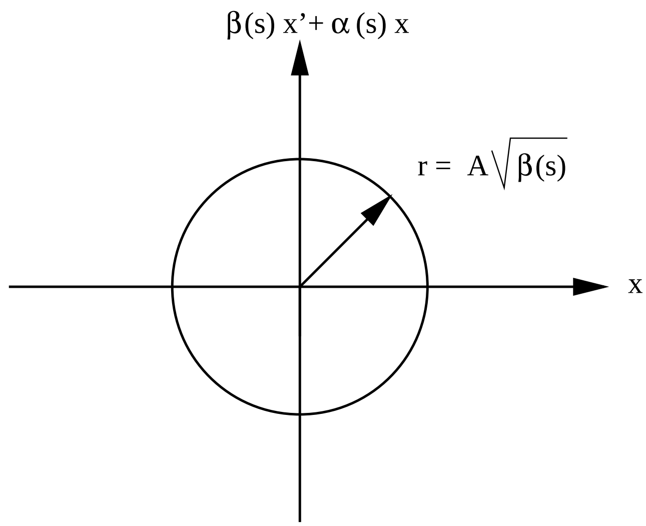

Figure 1: A circular trajectory in horizontal phase space of a particle at location s.

A circular phase space trajectory, such as that shown in Fig. 1 represents the evolution of the phase space coordinates of a particle with time, although at a specific location, s, in the machine. (If a beam makes a single pass through a machine, then there will only be one point for a given particle, no trajectory that evolves with time.) Since $\beta(s)$ is a function of s, the radius of the circular phase space trajectory for a given particle varies according to its location in the machine. There is a different phase space diagram for each machine location.

One use of circular phase space is that the distribution function is in a convenient form for integration to find a fractional beam emittance. See, for example, the calculation of 95% emittance at the end of this subsection (2.4). More typically the variables x, x' or x, p_x are used as phase space coordinates, since these are variables (closely related to the canonical variables) directly describing the motion. To find the form of the phase space trajectories in x, x' space, multiply Eq. 4 through and collect terms. This gives an equation for an ellipse in the x-x' plane:

$$\gamma x^2 + 2\alpha x x' + \beta (x')^2 = A^2 \tag{5}$$

where γ is defined as,

$$\gamma(s) \equiv \frac{(1 + \alpha(s)^2)}{\beta(s)}$$

An elliptical phase space trajectory is shown in Fig. 2. The general equation for an ellipse in the x-y plane is $ax^2 + 2bxy + cy^2 = d$. The area of this ellipse is Area$= \frac{\pi d}{\sqrt{ac-b^2}}$. Using the expression for area (see Exercise 4 of the the transverse emittance tutorial) we find that the area enclosed by the particle trajectory of Eq.(5) is Area$= \pi A^2$, a constant.

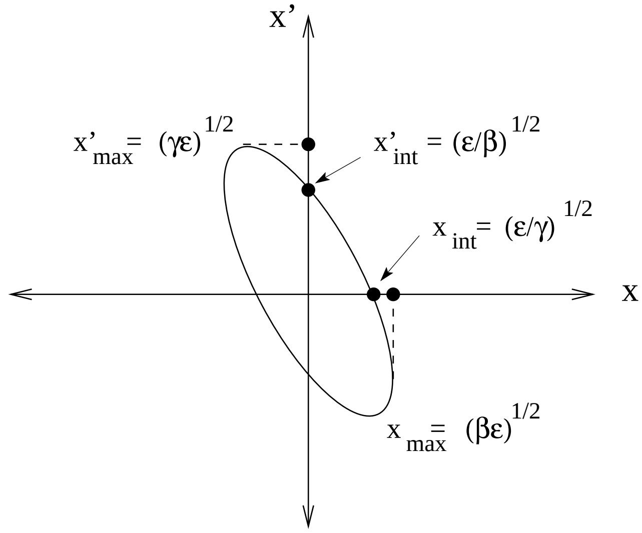

Figure 2: An elliptical trajectory in horizontal phase space of a particle at location s.

We have not yet obtained a beam emittance, since the area we have found is that enclosed by the motion of a single particle of amplitude $r = A\sqrt{\beta}$. Each particle in the beam follows its own phase space trajectory. In a particle accelerator, the particles comprising the beam have trajectories that do not cross each other; but still, a suitable trajectory must be chosen to define the emittance, ε. One typical choice is $r = \sigma$, where σ is the standard deviation of particle position in the beam. The area of the phase space ellipse enclosed by a particle trajectory was found to be πA^2. Sometimes, the emittance is defined to be this area, $\varepsilon = \pi A^2$, whereas other times it is defined excluding the π, in other words, $\varepsilon = A^2$. Choosing the latter, and considering the area enclosed at $r = \sigma$, we have $\sigma = A\sqrt{\beta} = \sqrt{\varepsilon\beta}$, or:

$$\varepsilon = \frac{\sigma^2}{\beta} \tag{6}$$

The substitution of $\varepsilon \equiv A^2$ in Eq. 5 is made to get the intercepts and maximum values of Fig. 2. To find the x$'$-intercept, set $x = 0$ in Eq. 5 and solve for x' (similarly for the x-intercept). Finding the maximum values for x and x$'$ is a little more work. For example, to find x$'_{max}$, first solve Eq. 4 for x$'$ in terms of x (with $A^2 = \varepsilon$) to get $x'(x) = \frac{\sqrt{\varepsilon\beta - x^2} - \alpha x}{\beta}$. Put the resulting expression for $x'(x)$ into $\frac{dx'}{dx} = 0$ and solve for x; this is the value of x at x$'_{max}$. This value, $x = \frac{-\alpha\sqrt{\varepsilon\beta}}{\sqrt{1+\alpha^2}}$, can then be substituted into Eq. 4, and solving the equation for x$'_{max}$ yields the solution $x'_{max} = \sqrt{\gamma\varepsilon}$.

If instead, the former choice for ε including the π is made, the expression for emittance is the following:

$$\varepsilon = \frac{\pi\sigma^2}{\beta} \tag{7}$$

When using the expression for emittance given by Eq. 7, people often quote 'π mm-mrad' as the units of emittance, leaving the π explicitly stated rather than multiplying through by its numerical value. The mm-mrad specifies that the emittance is on the order of 10^{-6}. For example, 30 π mm-mrad would be an emittance of $[30(3.141)] \times 10^{-6}$ meter-radians. The expression for emittance is not the only thing that has no uniform convention; the units can also be differently specified under different circumstances. As for example, emittances of electron beams upon emission are often given as μm, say 0.1 μm, indicating 0.1×10^{-6} meter-radians.

Beware: even if the x and y motion is not coupled, so that the simple expression of Eq. 6 can be used, make sure to put the appropriate β and σ in for each:

$$\varepsilon_x = \frac{\sigma_x^2}{\beta_x(s)} \qquad\qquad \varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)}$$

Notice that in these expressions for emittance, there is no explicit presence of the x' coordinate. In a well-behaved high energy machine, the motion of particles is governed by the β function. Knowledge of the β function at a detector location allows the emittance to be determined from measurement of the beam sigma there. At locations where β is large, the beam size (σ) goes up accordingly. In the horizontal plane, there is an additional complication due to magnet dispersion. A spread in the momentum of the particles of the beam will spread out the particle positions. A correction to the expression for horizontal emittance is needed to take this into account. Then:

$$\varepsilon_x = \frac{\sigma_x^2}{\beta_x(s)} - \frac{D(s)^2}{\beta_x}\left(\frac{\sigma_p}{p_0}\right)^2$$

where $D(s)$ is the dispersion at location s, p_0 is the ideal particle momentum, and σ_p is the root-mean-square of the momentum difference of the particles in the beam from the ideal momentum.

Now, suppose instead of defining the emittance to be the area enclosed by the contour at $r = \sigma$, we want to define it to be the contour enclosing 95% of the phase space area [1].

We can find the appropriate expression for the emittance directly by integrating over an appropriate distribution function, as in Eq. 1, but here using the circular phase space variables.

$$dN \quad = \quad f(x, y) \;\; dx \;\; dy$$

$$
\begin{aligned}
&= f(x, \beta x' + \alpha x) \;\; d(\beta x' + \alpha x) \;\; dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_x} \;\; e^{-\frac{x^2}{2\sigma_x^2}} \left[\frac{1}{\sqrt{2\pi}\sigma_{\beta x' + \alpha x}} \;\; e^{-\frac{(\beta x' + \alpha x)^2}{2(\sigma_{\beta x' + \alpha x})^2}} \;\; d(\beta x' + \alpha x) \right] dx
\end{aligned} \tag{8}
$$

As at the end of subsection 2.1, dN represents the fraction of the total number of particles. To take advantage of the geometry of the circular phase space trajectories, r', θ coordinates will be used, with the radial variable of integration defined as $(r')^2 \equiv [(\beta x' + \alpha x)^2 + x^2]$. Since there is circular symmetry, $\sigma_x = \sigma_{\beta x' + \alpha x}$ Then:

$$
\begin{aligned}
.95 &= \left(\frac{dN}{N}\right)_{95} = \int_0^{2\pi} \int_0^{r_{95}} f(r', \theta) r' dr' d\theta \\
&= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{r^{95}} e^{-\frac{(r')^2}{2\sigma^2}} \;\; r' dr' \\
&= \frac{1}{\sigma^2} \frac{1}{2} (-2\sigma^2) \left[e^{-\frac{r_{95}^2}{2\sigma^2}} - 1 \right]
\end{aligned}
$$

Solving for r_{95}^2:

$$
r_{95}^2 = -2\sigma^2 \ln{(.05)}
$$

Substituting $r^2 = A^2\beta = \frac{\varepsilon\beta}{\pi}$:

$$
\varepsilon_{95} = \frac{6\pi\sigma^2}{\beta}
$$

2.5 Normalized emittance

When a beam accelerates, the transverse beam size shrinks. The idea of invariant emittance can still be used if the emittance is scaled according to the beam energy, $\varepsilon_N = \varepsilon(\beta\gamma)$. Here β and γ are the relativistic parameters defined by the beam energy. The 'normalized emittance', ε_N, is then constant as a beam changes energy. A qualitative idea of the source of the shrinking beam can be seen in figure 3.

The transverse angles before and after are given by the following:

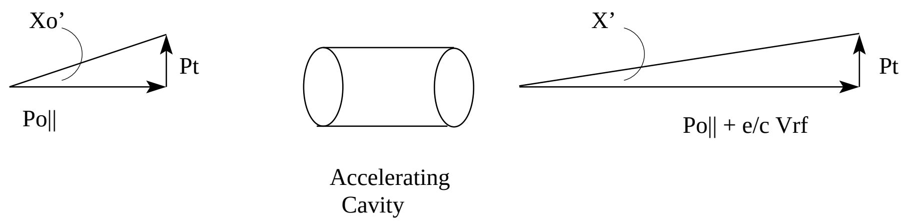

Figure 3: Cartoon of x$'$ shrink due to acceleration.

$$X_0' = \frac{P_t}{P_{0||}}$$

$$X' = \frac{P_t}{P_{0||} + \frac{e}{c}V_{rf}} = \frac{P_t}{P_{||}}$$

Since $P_{0||} < P_{0||} + \frac{e}{c}V_{rf}$ then $X' < X_0'$, demonstrating heuristically that the size of the beam in transverse phase space shrinks with acceleration. Another way to understand the same thing is to solve Eq. 2 for X', that is,

$$X' = \frac{P_t}{m_0 c\beta\gamma} = \frac{P_t}{P_{total}}$$

For the case of a high energy beam, $P_{total} \approx P_{||}$, where $P_{||}$ in this example is the longitudinal momentum after the acceleration. Since X' scales as $\frac{1}{\beta\gamma}$ simply from acceleration, to recover an invariant emittance it is necessary to multiply by a factor of $\beta\gamma$

2.6 Statistical definition of emittance

There are circumstances when an emittance expression derived from Hill's equation, such as Eq. 6, may either not be accurate or else not easy to apply. An expression for emittance based purely on the distribution of particles in phase space is often more practical. Particle simulations, which have knowledge of the phase space coordinates of all particles in the beam, usually employ such a statistical definition for emittance. In addition, beam measurements made near a beam source are frequently set up to yield both position and angle information for emittance reconstruction.

The derivation of statistical emittance in this section follows the one given by Buon [2, 3]. Let the coordinates of a 2D phase space be position, w, and angle, w'. In general, the second-order moments of a distribution of N points on the w-w' plane are given by:

$$\sigma_w = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_i - < w >)^2} \qquad \sigma_{w'} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_i' - < w' >)^2}$$

Choose the coordinate axes w, $w^{'}$ so that the origin is at the barycenter of the phase space points. Then, the average position and angle of the particles in the distribution are zero ($< w >=< w^{'} >= 0$). This simplifies the expressions for the second order moments:

$$\sigma_w = \sqrt{\frac{1}{N}\sum_{i=1}^{N} w_i^2} \qquad \sigma_{w'} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_i')^2}$$

Orient the w-$w^{'}$ axes so that the sum of the squared distances of the phase space points to the w axis, σ_w, is minimized; while $\sigma_{w'}$ is maximized. The 'area', or emittance, of the w-$w^{'}$ distribution of points might then be defined as the characteristic width in w multiplied by the characteristic width in $w^{'}$:

$$\varepsilon = 2\sigma_w 2\sigma_{w'} = 4\sqrt{\sigma_w^2 \sigma_{w'}^2} \tag{9}$$

The 4 in Eq. 9 is optional. The reference coordinate axes of a measured or simulated particle distribution will not necessarily have the above convenient position and orientation. The distribution may be rotated so that there is a correlation between the position and angle coordinates of the points, or translated so that the mean values of the coordinates are not zero. It is desirable to write the expression for emittance (Eq. 9) in terms of the actual available coordinates. Suppose the measured or simulated 2D particle coordinates are called x and $x^{'}$. The ideal w-$w^{'}$ axes are rotated by an angle, θ, with respect to the x-$x^{'}$ axes. Assume for now that there is no translation, so that the origins of the coordinate axes are the same. The two sets of coordinates are related through the angle of rotation. The angle, θ, minimizes (maximizes) the mean square distance $(\sigma_w^{'})^2$ (σ_w^2) of the particle distribution (by the definition of the w-$w^{'}$ axes):

$$\frac{\partial(\sigma_w^{'})^2}{\partial\theta} = 0 \tag{10}$$

where,

$$(\sigma_w^{'})^2 = \frac{1}{N}\sum_{i=1}^{N}(d_i^{'})^2 \tag{11}$$

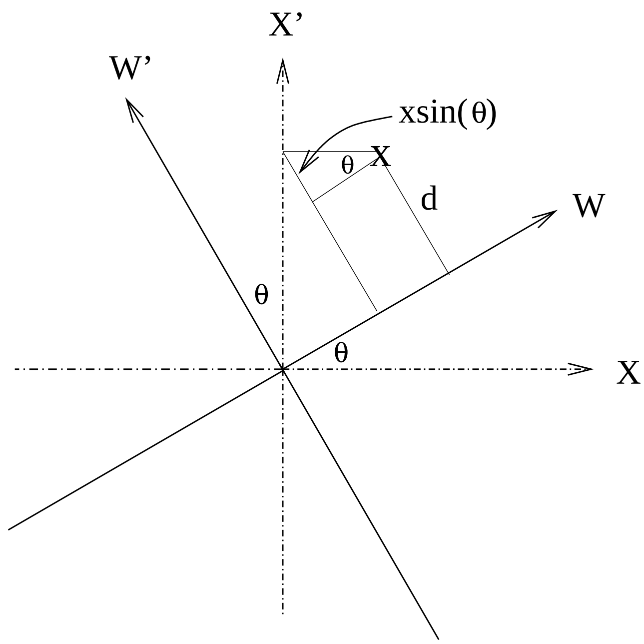

Figure 4: Rotation of phase space axes by angle θ. There is correlation of the x and x' variables of the particle distribution, but no correlation of the w and w' variables.

and d_i is the $w^{'}$ coordinate of the i^{th} particle. This is also the distance of the i^{th} phase space point from the w axis.

The distance, d_i of phase space point i to the $w^{'}$ axis, may be related to the phase space coordinates x_i, $x_i^{'}$ through the rotation angle θ. This relationship is illustrated in Fig. 4, and described by Eq. 12.

$$d_i^{'} = |x_i^{'} \cos(\theta) - x_i \sin(\theta)| = \sqrt{(x_i^{'} \cos(\theta) - x_i \sin(\theta))^2} \tag{12}$$

The distance of phase space point i to the w axis may also be related to the phase space coordinates x_i, $x_i^{'}$ through the rotation angle θ. This relationship can be seen in Fig. 4, and is described by Eq. 13.

$$d_i = |x_i^{'} \sin(\theta) + x_i \cos(\theta)| = \sqrt{(x_i^{'} \sin(\theta) + x_i \cos(\theta))^2} \tag{13}$$

Then, $(\sigma_w^{'})^2$ in terms of x_i and $x_i^{'}$ is the following:

$$
\begin{aligned}
(\sigma_w^{'})^2 &= \frac{1}{N}\sum_{i=1}^{N}(d_i^{'})^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}(x_i^{'}\cos(\theta) - x_i\sin(\theta))^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}(x_i^{'})^2\cos^2(\theta) + \frac{1}{N}\sum_{i=1}^{N}x_i^2\sin^2(\theta) - \frac{1}{N}\sum_{i=1}^{N}2x_ix_i^{'}\sin(\theta)\cos(\theta) \\
&= <(x^{'})^2>\cos^2(\theta) + <x^2>\sin^2(\theta) - 2<xx^{'}>\sin(\theta)\cos(\theta) \\
&= \frac{1}{2}\Big[<(x^{'})^2>(1+\cos(2\theta)) + <x^2>(1-\cos(2\theta)) - 2<xx^{'}>\sin(2\theta)\Big] \qquad (14)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\sigma_w)^2 &= \frac{1}{N}\sum_{i=1}^{N}d_i^2 \\
&= \frac{1}{2}\Big[<(x^{'})^2>(1-\cos(2\theta)) + <x^2>(1+\cos(2\theta)) + 2<xx^{'}>\sin(2\theta)\Big] \qquad (15)
\end{aligned}
$$

The emittance $\varepsilon = 4\sqrt{\sigma_w^2\sigma_{w'}^2}$ may now be written in terms of $<x^2>$, $<(x^{'})^2>$, $<xx^{'}>$ and angle θ. The minimization condition, Eq. 10 may be used to get rid of the explicit θ dependence, leaving an equation for the emittance that depends only on the phase space variables x and $x^{'}$. Plugging Eq. 14 into Eq. 10:

$$
\begin{aligned}
\frac{\partial(\sigma_{w'})^2}{\partial\theta} &= \frac{1}{2}\frac{\partial}{\partial\theta}\Big[<(x^{'})^2>(1+\cos(2\theta)) + <x^2>(1-\cos(2\theta)) - 2<xx^{'}>\sin(2\theta)\Big] \\
&= -<(x^{'})^2>(\sin(2\theta)) + <x^2>(\sin(2\theta)) - 2<xx^{'}>\cos(2\theta) \\
&= 0 \qquad (16)
\end{aligned}
$$

Using Eq. 16, x and $x^{'}$ are related to the angle, θ, as follows:

$$\tan 2\theta = \frac{2 < xx^{'} >}{< x^2 > - < (x^{'})^2 >} \tag{17}$$

Expressions for the sine and cosine are obtained directly from Eq. 17:

$$\sin 2\theta = \frac{2 < xx^{'} >}{\sqrt{(< x^2 > - < (x^{'})^2 >)^2 + (2 < xx^{'} >)^2}} \tag{18}$$

$$\cos 2\theta = \frac{< x^2 > - < (x^{'})^2 >}{\sqrt{(< x^2 > - < (x^{'})^2 >)^2 + (2 < xx^{'} >)^2}} \tag{19}$$

Next, Eq. 18 and Eq. 19 can be plugged into Eq. 14 and Eq. 15 to get rid of the angle dependence. This results in the following equations, Eq. 20 and Eq. 21:

$$\begin{aligned} \sigma^2_{w'} &= \frac{1}{2}\left(< x^2 > + < (x^{'})^2 > - \frac{2 < xx^{'} >}{\sin(2\theta)}\right) \\ &= \frac{1}{2}\left(< x^2 > + < (x^{'})^2 > - \sqrt{(< x^2 > - < (x^{'})^2 >)^2 + (2 < xx^{'} >)^2}\right) \end{aligned} \tag{20}$$

$$\begin{aligned} \sigma^2_{w} &= \frac{1}{2}\left(< x^2 > + < (x^{'})^2 > + \frac{2 < xx^{'} >}{\sin(2\theta)}\right) \\ &= \frac{1}{2}\left(< x^2 > + < (x^{'})^2 > + \sqrt{(< x^2 > - < (x^{'})^2 >)^2 + (2 < xx^{'} >)^2}\right) \end{aligned} \tag{21}$$

So that the emittance can be written (leaving off the 4 in Eq. 9):

$$\begin{aligned} \varepsilon &= \sqrt{\sigma^2_w \sigma^2_{w'}} \\ &= \sqrt{\frac{1}{4}\left[(< x^2 > + < (x^{'})^2 >)^2 - (< x^2 > - < (x^{'})^2 >)^2 - (2 < xx^{'} >)^2\right]} \end{aligned}$$

$$= \sqrt{< x^2 >< (x')^2 > - < xx' >^2} \quad (22)$$

In the case where the mean values of the distribution of points in the $x - x^{'}$ phase space are not zero, $< x > \neq 0$ and $< x^{'} > \neq 0$, then:

$$\sigma_x^2 = \quad < x^2 >= \quad \frac{1}{N}\sum_{i=1}^{N}(x_i - < x >)^2$$

$$\sigma_{x'}^2 = \quad < (x')^2 >= \quad \frac{1}{N}\sum_{i=1}^{N}(x_i' - < x' >)^2$$

$$\sigma_x\sigma_{x'} = \quad < xx' >= \quad \frac{1}{N}\sum_{i=1}^{N}(x_i - < x >)(x_i' - < x' >)$$

Equation 22 may be written using a determinant:

$$\varepsilon = \sqrt{\begin{vmatrix} \sigma_x^2 & \sigma_x\sigma_{x'} \\ \sigma_{x'}\sigma_x & \sigma_{x'}^2 \end{vmatrix}}$$

Or, making the abbreviation $\sigma_x\sigma_x = \sigma_{xx}$, $\sigma_x\sigma_{x'} = \sigma_{xx'}$ and so forth, we have:

$$\varepsilon = \sqrt{\begin{vmatrix} \sigma_{xx} & \sigma_{xx'} \\ \sigma_{x'x} & \sigma_{x'x'} \end{vmatrix}}$$

$$= \sigma_x\sigma_{x'}\sqrt{1 - r^2}$$

where r is defined to be the correlation coefficient, $r = \frac{\sigma_x\sigma_{x'}}{\sqrt{\sigma_x^2(\sigma_{x'})^2}}$, whose absolute value is less than or equal to one. The correlation coefficient will be zero in the absence of correlation of the phase space variables, x and x'.

The full expression for emittance in 6D phase space is the following:

$$\varepsilon^2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xx'} & \sigma_{xy} & \sigma_{xy'} & \sigma_{xz} & \sigma_{xz'} \\ \sigma_{x'x} & \sigma_{x'x'} & \sigma_{x'y} & \sigma_{x'y'} & \sigma_{x'z} & \sigma_{x'z'} \\ \sigma_{yx} & \sigma_{yx'} & \sigma_{yy} & \sigma_{yy'} & \sigma_{yz} & \sigma_{yz'} \\ \sigma_{y'x} & \sigma_{y'x'} & \sigma_{y'y} & \sigma_{y'y'} & \sigma_{y'z} & \sigma_{y'z'} \\ \sigma_{zx} & \sigma_{zx'} & \sigma_{zy} & \sigma_{zy'} & \sigma_{zz} & \sigma_{zz'} \\ \sigma_{z'x} & \sigma_{z'x'} & \sigma_{z'y} & \sigma_{z'y'} & \sigma_{z'z} & \sigma_{z'z'} \end{vmatrix}$$

In the absence of correlation between the phase planes, the horizontal, vertical, and longitudinal motion may be treated independently. Many of the off-diagonal terms are zero, reducing the 6X6 determinant to three 2X2 determinants, one for each plane.

$$\varepsilon^2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xx'} & 0 & 0 & 0 & 0 \\ \sigma_{x'x} & \sigma_{x'x'} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{yy} & \sigma_{yy'} & 0 & 0 \\ 0 & 0 & \sigma_{y'y} & \sigma_{y'y'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{zz} & \sigma_{zz'} \\ 0 & 0 & 0 & 0 & \sigma_{z'z} & \sigma_{z'z'} \end{vmatrix}$$

It is possible to write the statistically measurable parameters, σ_x and $\sigma_{x'}$ in terms of the Twiss parameters and the emittance. Thereby an effective β, α, and γ may be defined. This is done by comparing the elliptical trajectory from the Hill's equation analysis to an ellipse defined by semi-major and semi-minor axes σ_w and $\sigma_{w'}$ (it doesn't matter which is which).

The Hill's ellipse was found to be:

$$\gamma x^2 + 2\alpha x x' + \beta (x')^2 = \varepsilon$$

While the ellipse in w-w' space is given by:

$$\frac{w^2}{\sigma_w^2} + \frac{(w')^2}{\sigma_{w'}^2} = 1$$

The equation for the ellipse in w-w' space must be transformed to x-x' space by an inverse rotation. Once that is done, the two equations may be directly compared to find the desired relationships that relate σ_x and $\sigma_{x'}$ to the Twiss parameters. These are;

$$\sigma_x = \sqrt{\varepsilon\beta}$$

$$\sigma_{x'} = \sqrt{\varepsilon\gamma}$$

$$r\sigma_x\sigma_{x'} = -\alpha\varepsilon$$

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