Holographic entanglement entropy and modular Hamiltonian in warped CFT in the framework of GMMG model

M. R. Setare
Based of EPJC (2021)
in collaboration with
M. Koohgard

Outline

- Introduction and motivation
- Warped CFT
- Modular flow in WCFT
- Entanglement in WCFT
- The first law of entanglement entropy in WCFT
- Warped AdS black hole solution of GMMG
- The first law of entanglement entropy in $WAdS_3$
- Summary

Introduction and motivation

- There are many motivation for studying gravity in 3 dimensions.
- By this study we can address conceptual issues of quantum gravity, investigate black hole evaporation, information loss and black hole microstate counting.
- Also we can understand the black hole holography deeper.
- Gauge gravity duality can be extend to standard AdS/CFT, such as warped AdS, asymptotic Schrodinger/Lifshitz nonrelativistic CFTs, flat space holography, logarithmic CFTs and higher spin gravity.
- It is well known that topological massive gravity (TMG) and new massive gravity (NMG) in 3D have a bulk-boundary unitary conflict.
- Either the bulk or the boundary theory is non-unitary.

- There is a clash between the positivity of the two Brown-Henneaux boundary C charges and the bulk energies.
- Bergshoff et al. proposed a new model named minimal massive gravity (MMG), which has the same minimal local structure as TMG. [CQG (2014)]
- The single massive degree of freedom of MMG is unitary in the bulk and gives rise to a unitary CFT on the boundary.
- Following this paper, we have introduced generalized minimal massive gravity (GMMG). [M.R.S., NPB (2015)]
- GMMG is realized by adding higher-derivative deformation term to the Lagrangian of MMG.
- GMMG also avoids the aforementioned “bulk-boundary” unitary clash.
- Hamiltonian analysis shows that GMMG model has Boulware-Deser ghost and this model propagates only two physical modes.

- The entanglement entropy and the modular Hamiltonian have important roles in establishment of WAdS/WCFT duality and in verifying the first law of entanglement entropy.
- Equality of the gravitational charge related to the isometries of the warped AdS_3 spacetime and the modular Hamiltonian of WCFT is an evidence to the holography conjecture.

Warped CFT

- We consider a two dimensional theory

  defined on a plane with coordinates (z, w).
- The spacetime symmetries are generated by the following transformations

$$z' = f(z'), \quad w' = w + g(z') \qquad (*)$$

  $f(z)$ and $g(z)$ are two arbitrary functions.
- The theories which are invariant under the transformations (*) are known as WCFTs.
- The warped conformal transformation (*) is generated by two operators $T(z)$

  and $P(z)$.

  $T(z)$ is generator of infinitesimal coordinate transformation in z ,

  $P(z)$ is generator of z-dependent infinitesimal coordinate translations in w .

- We can define the charges on the plane as follows

$$
\begin{aligned}
L_n &= -\frac{i}{2\pi}\int dz z^{n+1} T(z), \\
P_n &= -\frac{1}{2\pi}\int dz z^{n} P(z).
\end{aligned}
$$

These charges satisfy a Virasoro-Kac-Moody algebra with central charge c and $U(1)$ level k as follows

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}, \\
[P_n, P_m] &= \frac{k}{2}n\delta_{n+m}, \\
[L_n, P_m] &= -mP_{n+m},
\end{aligned}
$$

[D. M. Hofman and A. Strominger, PRD 2011]

- We are interested to find a mapping from the plane to the cylinder in coordinate (z', w').

- The changes in the coordinates are as follows

$$z = e^{-iz'}, \qquad w = w' + 2\alpha z'$$

Then

$$\begin{aligned} P^{cyl}(z') &= izP(z) - k\alpha & (*) \\ T^{cyl}(z') &= -z^2T(z) + \frac{c}{24} + 2i\alpha zP(z) - k\alpha^2 & (**) \end{aligned}$$

- The modes on the cylinder can be defined as follows

$$\begin{aligned} L_n^{cyl} &= -\frac{1}{2\pi}\int dz' e^{inz'} T^{cyl}(z') \\ P_n^{cyl} &= -\frac{1}{2\pi}\int dz' e^{inz'} P^{cyl}(z') \end{aligned}$$

Substituting Eqs. (*) and (**) in the charges on the cylinder, we find

$$\begin{aligned} P_n^{cyl} &= P_n + \alpha k \delta_{n,0} \\ L_n^{cyl} &= L_n + 2\alpha P_n + (\alpha^2 k - \frac{c}{24})\delta_{n,0} \end{aligned}$$

- We can find the vacuum charges on the cylinder using the above equations

$$P_0^{vac} \equiv \langle P_0^{cyl} \rangle = \alpha k$$
$$L_0^{vac} \equiv \langle L_0^{cyl} \rangle = \alpha^2 k - \frac{c}{24}$$

- Modified algebra is as

$$\left[\tilde{L}_n, \tilde{L}_m\right] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m},$$
$$\left[\tilde{P}_n, \tilde{P}_m\right] = 2n\tilde{P}_0\delta_{n+m},$$
$$\left[\tilde{L}_n, \tilde{P}_m\right] = -m\tilde{P}_{n+m} + m\tilde{P}_0\delta_{n+m},$$

[Detournay, Hartman, Hofman, PRD 2012]

- We could return to the original algebra by the following re-definition of the charges

$$\tilde{P}_n = \frac{2}{k}P_0P_n - \frac{1}{k}P_0^2\delta_n,$$
$$\tilde{L}_n = L_n - \frac{2}{k}P_0P_n + \frac{1}{k}P_0^2\delta_n$$

Modular flow in WCFT

- We find the modular flow generator associated with a single interval in WCFT from the generalized Rindler transformation.
- We consider an interval $\mathcal{I}$ on the vacuum state as follows

$$\partial\mathcal{I} = \{(z_-, w_-), (z_+, w_+)\};$$

$$l_z = z_+ - z_-, \quad l_w = w_+ - w_-.$$

where (z, w) are the coordinates on the plane .

- The symmetry generators $SL(2,R) \times U(1)$ that leave the causal domain $\mathcal{I}$ invariant are as follows

$$\begin{aligned} l_n &= -z^{n+1}\partial_z; \qquad n = -1, 0, +1 \\ \bar{l}_0 &= -\partial_w, \end{aligned}$$

where $\bar{l}_0$ is $U(1)$ generator; l_n are the $SL(2,R)$ generators as follows

$$\begin{aligned} [l_-, l_+] &= 2l_0 \\ [l_0, l_\pm] &= \pm l_\pm. \end{aligned}$$

- The modular flow generator as a linear combination of the vacuum symmetry generators can be written as follows

$$\zeta = \zeta^z \partial_z + \zeta^w \partial_w = \Sigma_{j=-1}^{1} a_j l_j + \bar{a}_0 \bar{l}_0.$$

- To find the coefficients a_i , it is needed the modular flow ζ vanishes at the boundaries of the causal domain.
  We find the coefficients as follows

$$(a_+, a_0, a_-) = \frac{2\pi}{z_+ - z_-}(1, -(z_+ + z_-), z_+ z_-)$$

$$\bar{a}_0 = 2\pi\mu$$

- Substituting all the coefficients into above ζ , we find the modular flow generator in the interval as follows

$$\begin{aligned} \zeta &= 2\pi\mu\bar{l}_0 + \frac{2\pi}{z_+ - z_-}(l_1 - (z_+ + z_-)l_0 + z_+ z_- l_{-1}) \\ &= -2\pi\mu\partial_w - \frac{2\pi}{z_+ - z_-}(z_+ z_- - (z_+ + z_-)z + z^2)\partial_z \end{aligned}$$

This is calculated in WCFT using the generalized Rindler method

Entanglement entropy in WCFT

- To calculate the entanglement entropy of a single interval, the background geometry is considered as a cylinder with coordinates (t, x) .

The interval domain will be defined as follows

$$\mathcal{D} : (t, x) \in \left[(\frac{\bar{l}}{2}, -\frac{l}{2}), (-\frac{\bar{l}}{2}, \frac{l}{2}) \right]$$

- The entanglement entropy in $\mathcal{D}$ using the density matrix $\rho_{\mathcal{D}}$ is defined as follows

$$S_{EE} = -Tr(\rho_{\mathcal{D}} \log \rho_{\mathcal{D}})$$

- Using a unitary transformation, the entanglement entropy is related to a thermal entropy.

- For a warped system, the only allowed transformation is as follows

$$\frac{\tan\frac{\pi x}{L}}{\tan\frac{\pi l}{2L}} = \tanh\frac{\pi\tilde{x}}{L}, \quad t + \frac{\bar{L}}{L}x = \tilde{t} + \frac{\bar{\kappa}}{\kappa}\tilde{x}$$

where κ and $\bar{\kappa}$ are arbitrary scales and induces the following identification in $(\tilde{t}, \tilde{x})$ coordinates

$$\mathcal{H} : (\tilde{t}, \tilde{x}) \sim (\tilde{t} - i\bar{\kappa}, \tilde{x} + i\kappa)$$

- The thermal density matrix of the domain $\mathcal{H}$ is related to the density matrix $\rho_{\mathcal{D}}$ as follows

$$\rho_{\mathcal{D}} = U \rho_{\mathcal{H}} U^{\dagger}$$

Then

$$S_{EE} = -Tr(\rho_{\mathcal{D}} \log \rho_{\mathcal{D}}) = S_{thermal}(\mathcal{H})$$

To evaluate the thermal entropy, it is needed to find the partition function for $\mathcal{H}$

$$Z_{\bar{a}|a}(\bar{\theta}|\theta)$$

where $(a, \bar{a})$ and $(\theta, \overline{\theta}\ )$ are defined by the following identifications

$$(\tilde{t}, \tilde{x}) \sim (\tilde{t} + 2\pi\bar{a}, \tilde{x} - 2\pi a) \sim (\tilde{t} + 2\pi\bar{\theta}, \tilde{x} - 2\pi\theta)$$

The thermal entropy is defined as follows

$$S_{\bar{a}|a}(\bar{\theta}|\theta) = (1 - \theta\partial_\theta - \bar{\theta}\partial_{\bar{\theta}}) \log Z_{\bar{a}|a}(\bar{\theta}|\theta)$$

The partition function can be calculated as follows

$$\hat{Z}(\rho, \tau) = e^{i\pi\frac{k}{2}\frac{\rho^2}{\tau}} \hat{Z}(\frac{\rho}{\tau}| -\frac{1}{\tau}) = e^{i\pi\frac{\kappa}{2}\frac{\rho^2}{\tau}} e^{2\pi i\frac{\rho}{\tau}P_0^{vac} + 2\pi i\frac{1}{\tau}L_0^{vac}} + \ldots$$

[Castro, Hofman, Iqbal, JHEP 2016]

where

$$\tau = -i\frac{\pi}{\gamma}, \quad \rho = -\frac{i}{2\gamma}(\frac{\bar{L}}{L}l - \bar{l})$$

- Then, we find the entanglement entropy as follows

$$S_{EE} = iP_0^{vac} l(\frac{\bar{L}}{L} - \frac{\bar{l}}{l}) - 4L_0^{vac} \log(\frac{L}{\pi\epsilon} \sin\frac{\pi l}{L})$$

- To evaluate the entanglement entropy of one segment in the WCFT at finite temperature, all we need to do is changing the map that we used before, as follows

$$\frac{\tanh\frac{\pi x}{L}}{\tanh\frac{\pi l}{2L}} = \tanh\frac{\pi\tilde{x}}{L}, \quad t + \frac{\bar{\beta}}{\beta}x = \tilde{t} + \frac{\bar{\kappa}}{\kappa}\tilde{x}$$

- So, we just to use the replacement $L \to i\beta$ and $\bar{L} \to i\bar{\beta}$.

With this replacement, we find

$$S_{EE} = iP_0^{vac} l(\frac{\bar{\beta}}{\beta} - \frac{\bar{l}}{l}) - 4L_0^{vac} \log(\frac{\beta}{\pi\epsilon} \sinh\frac{\pi l}{\beta})$$

The first law of entanglement entropy in WCFT

- The modular Hamiltonian has the following relation with the partition function

$$\begin{aligned}\mathcal{H}_\zeta &= 2\pi i\rho P_0^{cyl} - 2\pi i\tau L_0^{cyl} - \log Z \\ &= -\frac{1}{2\pi}\int d\hat{x}\{2\pi i\rho\hat{P}(\hat{x}) - 2\pi i\tau\hat{T}(\hat{x})\} - \log Z\end{aligned}$$

where we use the L_n^{cyl} and P_n^{cyl} definitions on the cylinder.

- $\mathcal{H}_\zeta$ is the charge related to the modular flow generator vector ζ.
- The modular Hamiltonian is given by

$$\mathcal{H}_{mod} = \mathcal{H}_\zeta + const$$

[Aplo, Jiang, Song, Zhong, JHEP (2020)]

- Then, we find

$$\mathcal{H}_{mod} = \int_{x_-}^{x_+} dx \Big\{ 2i\alpha P(x) + \frac{\beta}{2\pi} \Big[\frac{\cosh\frac{\pi(2x-l)}{\beta} - \cosh\frac{\pi l}{\beta}}{\sinh\frac{\pi l}{\beta}} \Big] \Big[T(x) - \frac{\bar{\beta}}{\beta} P(x) \Big] \Big\}$$

$$+ \; ik\alpha\bar{l} - \Big[\frac{c}{12} - \frac{k}{8\pi^2}\bar{\beta}^2\Big]\Big[\frac{\pi l}{\beta}\cosh\frac{\pi l}{\beta} - 1\Big]$$

- The variation of the modular Hamiltonian can be find as follows

$$\delta\mathcal{H}_{mod} = \int_{x_-}^{x_+} dx \Big\{ 2i\alpha\delta P(x) + \frac{\beta}{2\pi} \Big[\frac{\cosh\frac{\pi(2x-l)}{\beta} - \cosh\frac{\pi l}{\beta}}{\sinh\frac{\pi l}{\beta}} \Big] \Big[\delta T(x) - \frac{\bar{\beta}}{\beta} \delta P(x) \Big]$$

$$+ \; \frac{ik\alpha}{l}\delta\bar{l} \Big\}$$

- Using L_n^{cyl} and P_n^{cyl} definitions on the cylinder, we find

$$\delta T(x) = -\delta L_0^{cyl},$$
$$\delta P(x) = -\delta P_0^{cyl}.$$

Then we find

$$\delta\mathcal{H}_{mod} = i\alpha k\delta\bar{l} - 2i\alpha l\delta P_0^{cyl} - \frac{\beta^2}{4\pi^2}\Big(2 - \frac{2\pi l}{\beta}\coth\frac{\pi l}{\beta}\Big)\Big(\delta L_0^{cyl} - \frac{\bar{\beta}}{\beta}\delta P_0^{cyl}\Big)$$

$$= iP_0^{vac}\delta\bar{l} - 2i\alpha l\delta P_0^{cyl} - \frac{\beta^2}{4\pi^2}\Big(2 - \frac{2\pi l}{\beta}\coth\frac{\pi l}{\beta}\Big)\Big(\delta L_0^{cyl} - \frac{\bar{\beta}}{\beta}\delta P_0^{cyl}\Big)$$

- The first law of entanglement entropy is a statement on the equality of the entanglement entropy variations with the modular Hamiltonian variations as follows

$$\delta S_{EE} = \delta\langle\mathcal{H}_{mod}\rangle$$

- The variation of the entanglement entropy can be found as follows

$$\delta S_{EE} = iP_0^{vac} l\delta(\frac{\bar\beta}{\beta}) - iP_0^{vac}\delta\bar l - 4L_0^{vac}(1-\frac{\pi l}{\beta}\coth\frac{\pi l}{\beta})\frac{\delta\beta}{\beta}$$

- Substituting the following relations in δS_{EE}

$$\begin{aligned}\frac{\delta\beta}{\beta^3} &= -\frac{3}{c}\frac{1}{\pi^2}\delta(L_0^{cyl} - \frac{(P_0^{cyl})^2}{k}),\\ \delta(\frac{\bar\beta}{\beta}) &= \frac{2}{k}\delta P_0^{cyl},\end{aligned}$$

[Aplo, Jiang, Song, Zhong, JHEP (2020)]

We find

$$\delta S_{EE} = iP_0^{vac}\delta\bar l - 2i\alpha l\delta P_0^{cyl} - \frac{\beta^2}{4\pi^2}(2-\frac{2\pi l}{\beta}\coth\frac{\pi l}{\beta})(\delta L_0^{cyl} - \frac{\bar\beta}{\beta}\delta P_0^{cyl})$$

- Comparing δS_{EE} and $\delta\langle\mathcal{H}_{mod}\rangle$, we find these results are equal to each other. This is a proof of the first law of the entanglement entropy .
- we can introduce the following functions $K(x)$ and $Y\ (x)$ into δS_{EE} relation

$$K(x) = \left(\frac{\bar{\beta}}{\beta}\right)\left[iP_0^{vac} f(l, P_0^{vac}, P_0^{cyl})\left(\frac{\beta}{\bar{\beta}}\right) - 2i\alpha l\left(\frac{\beta}{\bar{\beta}}\right) + \frac{\beta^2}{4\pi^2}\left(2 - \frac{2\pi l}{\beta}\coth\frac{\pi l}{\beta}\right)\right]$$

$$Y(x) = -\frac{\beta^2}{4\pi^2}\left(2 - \frac{2\pi l}{\beta}\coth\frac{\pi l}{\beta}\right)$$

Then we have

$$\delta S_{EE} = K(x)\delta P_0^{cyl} + Y(x)\delta L_0^{cyl}.$$

Substituting the cylinder charges into above equation, we find

$$\delta S_{EE} = -\frac{1}{2\pi}\int dx\left\{K(x)\delta P + Y(x)\delta T\right\}$$

- We can find the variation of the entanglement entropy using the charges in the modified form, as

$$\delta S_{EE} = -\frac{1}{2\pi}\int dx\{\tilde{K}(x)\delta\tilde{P}(x) + \tilde{Y}(x)\delta\tilde{T}(x)\}$$

where

$$\delta\tilde{P}_0 = -\delta\tilde{P}$$
$$\delta\tilde{L}_0 = -\delta\tilde{T}$$

and

$$\tilde{K}(x) = \frac{K(x)k + 2P_0^{cyl}Y(x)}{2(P_0^{cyl} - \alpha k)}$$

$$\tilde{Y}(x) = Y(x)$$

Warped AdS black holes solution of GMMG

- Generalized Minimal Massive Gravity (GMMG) provides a new example of a theory that avoids the bulk-boundary clash.
- So this model possesses positive energy excitations around the maximally AdS_3 vacuum as well as a positive central charge in the dual CFT .
- The lagrangian of GMMG model is as

$$L_{GMMG} = L_{GMG} - \frac{\alpha}{2} e.h \times h$$

where

$$L_{GMG} = L_{TMG} - \frac{1}{m^2}\left(f.R + \frac{1}{2} e.f \times f\right)$$

$$L_{TMG} = -\sigma e.R + \frac{\Lambda_0}{6} e.e \times e + h.T(\omega) + \frac{1}{2\mu}\left(\omega.d\omega + \frac{1}{3}\omega.\omega \times \omega\right)$$

Λ_0 is a cosmological parameter,
σ is a sign,
α is a dimensionless parameter,
e is a dreibein,
ω is dualized spin-connection,
f and h are auxiliary fields,
$T(\omega)$ and $R(\omega)$ are a Lorentz covariant torsion and a curvature 2-form, respectively.

- The warped AdS_3 black hole is a solution of the GMMG.

$$ds^2 = l^2\left(- N(r)^2 dt'^2 + \frac{dr^2}{4N(r)^2R(r)^2} + R(r)^2\left(d\phi + N^\phi(r)dt'\right)^2\right)$$

For the spacelike warped AdS_3 black hole we have

$$\begin{aligned} R(r)^2 &= \frac{1}{4}\alpha^2 r\left[(1-\nu^2)r + \nu^2(r_+ + r_-) + 2\nu\sqrt{r_+ r_-}\right] \\ N(r)^2 &= \alpha^2\nu^2\frac{(r-r_+)(r-r_-)}{4R(r)^2}, \\ N^\phi(r) &= |\alpha|\frac{r+\nu\sqrt{r_+ r_-}}{2R(r)^2}, \end{aligned}$$

where r_+ and r_- are the outer and inner horizon radiuses of the black hole, respectively.

- The appropriate boundary conditions to introduce asymptotically spacelike warped AdS_3 spacetime. is as

$$\begin{aligned}
g_{t't'} &= l^2, \quad g_{tr} = \mathcal{O}(r^{-3}), \quad g_{r\phi} = \mathcal{O}(r^{-2}), \\
g_{t'\phi} &= \frac{1}{2} l^2 |\alpha| \big[r + A_{t\phi}(\phi) + \frac{1}{r} B_{t\phi}(\phi) \big], \\
g_{rr} &= \frac{l^2}{\alpha^2 \nu^2} \big[\frac{1}{r^2} + \frac{1}{r^3} A_{rr}(\phi) + \frac{1}{r^4} B_{rr}(\phi) + \mathcal{O}(r^{-5}) \big], \\
g_{\phi\phi} &= \frac{1}{4} l^2 \alpha \big[(1-\nu^2) r^2 + r A_{\phi\phi}(\phi) + B_{\phi\phi}(\phi) \big] + \mathcal{O}(r^{-1})
\end{aligned}$$

[Hennaux, Martinez, Troncoso, PRD (2011)]

- The asymptotic Killing vectors that generate the fluctuations preserving the boundary conditions are as follows

$$\begin{aligned}
\xi^{t'}(\mathcal{K}, \mathcal{Y}) &= \mathcal{K}(\phi) - \frac{2\partial_\phi^2 \mathcal{Y}(\phi)}{|\alpha|^3 \nu^4 r} + \mathcal{O}(r^{-2}) \\
\xi^{r}(\mathcal{K}, \mathcal{Y}) &= -r \partial_\phi \mathcal{Y}(\phi) + \mathcal{O}(r^{-2}), \\
\xi^{\phi}(\mathcal{K}, \mathcal{Y}) &= \mathcal{Y}(\phi) + \frac{2\partial_\phi^2 \mathcal{Y}(\phi)}{\alpha^4 \nu^4 r^2} + \mathcal{O}(r^{-3})
\end{aligned}$$

$\mathcal{K}(\phi)$ and $\mathcal{Y}(\phi)$ are two arbitrary periodic functions.

- The corresponding conserved charge to the asymptotic Killing vectors is

$$\mathcal{Q}(\mathcal{K},\mathcal{Y}) = \mathcal{P}(\mathcal{K}) + \mathcal{L}(\mathcal{Y})$$

where

$$\mathcal{P}(\mathcal{K}) = -\frac{|\alpha|}{96\pi} c_U \int_0^{2\pi} \mathcal{K}(\phi)\big[A_{rr}(\phi) + 2A_{t\phi}(\phi)\big] d\phi$$

$$\mathcal{L}(\mathcal{Y}) = \frac{\alpha^4 \nu^4}{768\pi} c_V \int_0^{2\pi} \mathcal{Y}(\phi)\big[-3A_{rr}(\phi)^2 + 4B_{rr}(\phi) + 16B_{t\phi}(\phi)\big] d\phi$$

[M.R.S and Adami, Class. Quant. Grav. (2017)]

where

$$\begin{aligned} c_U &= \frac{3l|\alpha|\nu^2}{G}\left\{\sigma + \frac{\omega}{\mu}(H_1 + l^2 H_2) + \frac{1}{m^2}(F_1 + l^2 F_2) - \frac{|\alpha|}{2\mu l}\right\}, \\ c_V &= \frac{3l}{|\alpha|\nu^2 G}\left\{\sigma + \frac{\omega}{\mu}(H_1 + l^2 H_2) + \frac{1}{m^2}(F_1 + l^2 F_2) - \frac{|\alpha|}{2\mu l}(1 - 2\nu^2)\right\} \end{aligned}$$

H_1, H_2, F_1 and F_2 are constant parameters,

ω is a dimensionless parameter.

- Introducing $M(\phi)$ and $J(\phi)$ as follows

$$\begin{aligned} M(\phi) &= A_{rr}(\phi) + 2A_{t\phi}(\phi), \\ J(\phi) &= -3A_{rr}(\phi)^2 + 4B_{rr}(\phi) + 16B_{t\phi}(\phi) \end{aligned}$$

We have

$$\mathcal{P}(\mathcal{K}) = -\frac{|\alpha|}{96\pi} c_U \int_0^{2\pi} \mathcal{K}(\phi) M(\phi) d\phi$$

$$\mathcal{L}(\mathcal{Y}) = \frac{\alpha^4 \nu^4}{768\pi} c_V \int_0^{2\pi} \mathcal{Y}(\phi) J(\phi) d\phi.$$

In the case of the warped black hole, we have

$$\begin{aligned} A_{rr} &= r_+ + r_-, \quad A_{t\phi} = \nu\sqrt{r_+ r_-}, \\ B_{rr} &= r_+^2 + r_-^2 + r_+ r_-, \quad B_{t\phi} = 0. \end{aligned}$$

- We define the Fourier modes as

$$\begin{aligned}\mathcal{P}_m &= \mathcal{Q}(e^{im\phi},0)=\mathcal{P}(e^{im\phi})\\ \mathcal{L}_m &= \mathcal{Q}(0,e^{im\phi})=\mathcal{L}(e^{im\phi})\end{aligned}$$

Using this definition, the charges can be found as follows

$$\mathcal{P}_m(\mathcal{K})=-\frac{|\alpha|}{96\pi}c_U\int_0^{2\pi}\mathcal{K}(\phi)M(\phi)e^{im\phi}d\phi$$

$$\mathcal{L}_m(\mathcal{Y})=\frac{\alpha^4\nu^4}{768\pi}c_V\int_0^{2\pi}\mathcal{Y}(\phi)J(\phi)e^{im\phi}d\phi$$

The charge algebra can be written as

$$\begin{aligned}[\mathcal{L}_n,\mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m}+\frac{c_V}{12}n(n^2-1)\delta_{n+m},\\ [\mathcal{P}_n,\mathcal{P}_m] &= -\frac{c_U}{12}\mathcal{P}_0\delta_{n+m},\\ [\mathcal{L}_n,\mathcal{P}_m] &= -m\mathcal{P}_{n+m}+\frac{m}{2}\mathcal{P}_0\delta_{n+m}.\end{aligned}$$

The charge algebra is the same as the modified that we have used in the field theory side.

The first law of entanglement entropy in $WAdS_3$

- We can write the gravitational charge of the GMMG as

$$
\begin{aligned}
Q(\mathcal{K},\mathcal{Y}) &= -\frac{|\alpha|}{96\pi}c_U\int_0^{2\pi}\mathcal{K}(\phi)M(\phi)d\phi \\
&+ \frac{\alpha^4\nu^4}{768\pi}c_V\int_0^{2\pi}\mathcal{Y}(\phi)J(\phi)d\phi
\end{aligned}
$$

then

$$
\begin{aligned}
\delta Q(\mathcal{K},\mathcal{Y}) &= -\frac{|\alpha|}{96\pi}c_U\int_0^{2\pi}\mathcal{K}(\phi)\delta M(\phi)d\phi \\
&+ \frac{\alpha^4\nu^4}{768\pi}c_V\int_0^{2\pi}\mathcal{Y}(\phi)\delta J(\phi)d\phi
\end{aligned}
$$

- We assume the first law of the entanglement entropy to be true in the gravity side of the $WAdS_3/WCFT$ in the GMMG case as in the Einstein gravity case.

[Song, Wen, Xu, JHEP (2017)], [Apolo, Jiang, Song, Zhong, JHEP (2020)]

- Modular Hamiltonian is the same as gravitational charge, since we assume the first law of entropy is true, we can write $\delta S_{HEE} = \delta Q$

we can complete the dictionaries between the fields in two sides of the duality.

- The algebra of the charges that generate the isometries of the warped AdS_3 black hole in GMMG has an equal form with the charge algebra in WCFT.
- This correspondence between the bulk and the boundary symmetry is an indication of the $WAdS_3/WCFT_2$ holography.
- We could find the following dictionary between the bulk and the boundary variables,

$$\begin{aligned} -\frac{1}{2\pi}\tilde{P} &= -\frac{|\alpha|}{96\pi}c_U M(\phi) \\ \frac{1}{2\pi}\tilde{T} &= -\frac{\alpha^4\nu^4}{768\pi}c_V J(\phi) \end{aligned}$$

- Substituting the dictionary variables into the gravitational charge, we find

$$\delta Q = \delta \langle \mathcal{H}_\varsigma \rangle$$

- Using the dictionary, the equality between the entanglement entropy in WCFT and the holographic entanglement entropy is also established easily.

Summary

- We studied one class of the non-AdS holographies.
- We studied the entanglement entropy and the first law in the $WAdS_3/WCFT$ correspondence in the context of GMMG.
- The asymptotic symmetries of the bulk side of the duality was the same as the vacuum symmetry of the field theory side.
- By computing the gravitational charge in the bulk and the modular Hamiltonian at the boundary, we succeeded to show the first law of the entanglement entropy.
- In the bulk, the spacelike warped AdS_3 black hole solution of GMMG was considered.
- The charge algebra of the $WCFT_2$ is a Virasoro-Kac-Moody algebra and this is the same as the asymptotic charge algebra of the bulk theory.