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THE EMITTANCE CONCEPT

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ABSTRACT

An informal descriptive account is first given of the emittance concept and its underlying physical basis. This is followed by a discussion of the connection between emittance and entropy, and a number of questions relating to problems of current interest concerning such topics as emittance growth and equipartition between different degrees of freedom are raised. Although no new results are obtained, it is hoped that the discussion may be helpful in the search for new insights. The paper differs from that presented at the conference, and contains ideas which arose in discussion with T P Wangler at Los Alamos after the conference.

INTRODUCTION

The term 'emittance' seems to have emerged in the early fifties, soon after the advent of alternating gradient focusing, and the general use of matrix techniques in accelerator design. It was preceded by the concept of 'admittance', defined in 1952 by Sigurgeirsson[1] for an alternating gradient synchrotron as

$$A = \iint \Omega(x, z)\,dx\,dz \tag{1}$$

"where Ω is the solid angle, within which the direction of motion for a particle has to fall if it is to remain in the synchrotron without striking the walls of the vacuum chamber." In his discussion, which is in the context of the synchrotron, z is the vertical direction and x is the radial distance from the equilibrium orbit, neglecting the small curvature. (Here we use y and x for these two quantities). He shows that for symmetry about the xy and zy planes (his notation) the x and z variables can be separated, and that the bounding contour in the two directions is elliptical, and further that the area of the ellipses does not vary along the length of the beam. In general the areas of the ellipses corresponding to the two planes are different. The extension to emittance, as a figure of merit defining a beam with particles having coordinates that would just fill such an ellipse is straightforward. Further extension of the concept to the longitudinal direction is not difficult as discussed below.

The emittance was soon seen to be a convenient figure of merit for ion sources, though it was immediately recognized that a single number cannot contain information about the distribution of points projected on the xx' or yy' planes. This difficulty is particularly evident when

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aberrations are present, so that the distribution is not elliptical. For distributions that are roughly elliptical, conventions such as defining the emittance as the area of the smallest ellipse containing say 95% of the points are convenient, if not elegant. In modern usage the emittance is the area of the appropriate area divided by π.

Before discussing further some of the complications associated with the idea of emittance, some general observations about the properties of ensembles of particles in a Hamiltonian system without collisions will be summarized.

LIOUVILLE'S THEOREM

This theorem applies to ensembles of particles moving in a conservative system, under the action of an external potential, plus that arising from the smoothed-out self-fields of the particles themselves. By 'smoothed-out' is meant that individual particle-particle collisions are excluded; only the average fields, which do not depend on the positions of the individual particles, are included. The theorem, which is simply proved from Hamilton's equations and continuity, states that the density in 6 dimensional phase-space in the neighbourhood of any chosen point remains constant. From this it follows that if a surface is drawn around a portion of the phase-space 'fluid', its volume remains invariant though its shape can vary.

It is this invariance that makes the emittance a useful concept, though it is important to distinguish properties associated with the overall distribution from those associated with its projections. First we note that the variables xx' and yy' associated with the transverse emittance are not canonical, so that xx' space is not strictly 'phase-space'. In paraxial approximation, however, $x' = p_x/p_z$, so that $\beta\gamma x'$ and x are canonical. It is readily shown that for a <u>linear</u> focusing system, in which x, y and z motion are <u>decoupled</u>, projected areas enclosing a fixed number of points (each corresponding to a particle) are conserved. Furthermore, shapes are conserved apart from rotation and stretching. In particular, ellipses remain elliptical. These features are <u>not</u> present in linear systems with coupling between x and y directions, (for example, skew quadrupoles, or quadrupole + solenoid). For linear systems with axial symmetry, such as solenoids or conventional magnetic lenses, the x and y motion can be decoupled by making a transformation to the Larmor frame, which rotates with angular velocity $\omega_L = -eB_z/2\gamma m_o$.

For monoenergetic beams, with uniform line density in the direction of propagation, the transverse emittance can be determined from a knowledge of the particle positions and velocities in a small slice of beam of length Δz. If, however, there is structure in the z-direction, as is the case in a linear accelerator or synchrotron, the particles in a complete bunch need to be considered. The length of the bunch is just one wavelength in a linac, or the circumference C divided by the operating harmonic number n in a synchrotron. Momenta can be measured with respect to a particle moving at a velocity $f\lambda$ or fC/n. Alternatively, particles can be specified in terms of their phase with respect to the accelerating field and its derivative, ϕ and $\dot{\phi}$. In such cyclic systems any particles leaving the front of the bunch are replaced by others entering at the rear.

In this discussion some elementary properties of emittance for linear systems have been summarized; before proceeding it is necessary to attempt a definition.

DEFINITIONS OF EMITTANCE

The progress of a beam can be represented in general by the evolution of a distribution of points in six-dimensional phase space. In most beams the system is conservative, (though a notable exception is an electron beam or very high energy proton beam where synchrotron radiation is emitted). The question now is how best to quantify this distribution in terms of a single number. A heuristic method is to take a value proportional to the minimum sized hyper-ellipsoid (or ellipse for projections involving one spatial co-ordinate) circumscribing a certain fraction of the points. More precise is a description in terms of moments. The r.m.s. emittance $\bar{\varepsilon}$ of a projection in the xx' plane may be defined as [2,3]

$$\bar{\varepsilon}^2 = [16]\left(<x^2><x'^2> - <xx'>^2\right) \qquad (2)$$

The factor 16 is introduced in ref. 2 but not in ref. 3. Its purpose is to ensure parity with the earlier alternative definition in which the area of uniformly populated ellipse in xx' space is taken as $\pi\varepsilon$. (See for example ref. 4).

The following points are well established:

1) For a linear system in which x, y and z motion is decoupled, the normalized emittance $\beta\gamma\bar{\varepsilon}$ is invariant. In the presence of space-charge there are a limited number of distributions which produce a uniform projected distribution with sharp edge, and hence provide a linear contribution to the focusing; the best known of these is that of Kapchinskij and Vladimirskij, (K-V), discussed further below. The envelope equation for such beams, known as the K-V equation, is, for a drifting unaccelerated beam,

$$\ddot{x}+k(x)x-\frac{\varepsilon_x^2}{x^3}-\frac{2K}{x+y}=0 \tag{3}$$

where k(x) represents the focusing force and K is the dimensionless perveance.

2) For beams with non-uniform charge density, so that the contribution to focusing is non-linear, the K-V envelope equations are still valid provided that a) x and y are interpreted as r.m.s. values, b) the distribution has elliptical symmetry[3].

3) The value of $\bar{\varepsilon}$ is not, however, invariant. Although the value $d\bar{\varepsilon}/dz$ for a given situation can be found, the way that $\bar{\varepsilon}$ varies cannot be determined.

Most of the discussion so far has been confined to linear systems with separable variables. When the variables are coupled there are further invariants, corresponding to Eq. 2 though these are considerably more complicated in form[5].

So far, precise definition has been given only of the projected emittance $\bar{\varepsilon}_x$ or $\bar{\varepsilon}_y$. For systems with axial symmetry 'radial emittance' is sometimes used, with definition similar to Eq. 2 but with r in place of x. In a linear system this appears quite logical, since the envelope equation is as Eq. 2 with r substituted for x and y. This is convenient for examining non-linear systems in which some form of spherical aberration is present, such as might arise from imperfect lenses or non-linear space-charge forces in a beam with non-uniform radial density distribution. (As an example see ref. 6, this conference). It must be emphasized, however, that in general, although $x^2 + y^2 = r^2$, it is not true that $x'^2 + y'^2 = r'^2$ unless the initial beam has no particles with angular velocity about the axis. The rr' emittance is thus useful for initially laminar beams, but is less so, for example, for beams where thermal velocities arising from cathodes or plasma ion sources are important.

Alternative definitions of emittance have been used including both x and y motion in cylindrical beams. Using a hydrodynamic approach rather than the usual optical one, Lee, Yu and Barletta[7] define it as

$$\bar{\varepsilon}^2=\bar{r}^2\left(\overline{x'^2+y'^2}-\bar{r}'^2\right) \tag{4}$$

and use this to calculate emittance growth in a converging beam where the deviation from ballistic trajectories arising from space-charge is small.

THE PROBLEM OF EMITTANCE GROWTH

Of great technical importance is the question of how the emittance of a beam grows as it passes down an accelerator or a beam transport system. Much progress has been made in this field since pioneering work at CERN and Brookhaven over 20 years ago, but the subject is far from completely understood. Modern computers have enabled many impressive computations and similations to be made, but it is still not clear whether any further physical insights await revelation. It is not the intention to review this very extensive work here, but merely to list some questions which seem not to be resolved. It is assumed that the reader is familiar with recent work of Hofmann, Reiser, Wangler and others and earlier work of Lapostolle concerning the concept of non-linear field energy and its conversion to particle energy in the process of emittance growth. (See for example ref. 8 and earlier references therein). Some questions to which there do not seem to be agreed answers are listed below. Some of these will be discussed further in the next section.

(1) A beam is fed into a long uniform focusing system. In the presence of space-charge, but not Coulomb collisions, is a final equilibrium with $v_z >> v_x, v_y$ ultimately attained?

(2) If so, is it possible other than by direct computer simulation to determine the final emittance $\tilde{\varepsilon}_f$ from the initial value $\tilde{\varepsilon}_i$ and the initial matching conditions?

(3) Does the velocity distribution eventually become Maxwellian in the absence of collisions?

(4) Can we make any estimate of the time to reach equilibrium in terms of ε_i, σ/σ_0 and matching conditions? (Here σ/σ_0 is the tune depression of the 'equivalent' K-V beam with the same value of $\tilde{\varepsilon}$).

(5) What happens in a periodic system? Does the beam evolve to a periodically fluctuating state with Maxwellian velocity distribution?

(6) In a beam with unequal transverse energies in the two directions parallel to the symmetry planes, under what conditions is equipartition achieved? What is the physical mechanism?

(7) As 6, but for bunches in a linac.

(8) Can we find a way of estimating these equipartition times?

It is not difficult to add further related questions to this list. In the next section the physical nature of emittance is discussed, in the hope that this may lead to some further insights into the above questions.

THE PHYSICAL NATURE OF EMITTANCE

Emittance may be regarded as a figure of merit for the quality of a particle beam; it is clearly related to brightness, a fundamental concept in light optics, and much discussed also in electron optics. (A typical paper in which these relationships may be seen is given as ref. 9). Although brightness is strictly defined locally, the overall brightness of a particle beam is often defined as proportional to the current divided by the product of ε_x and ε_y. In both light optics and charged-particle optics there is a tendency for the quality of a transmitted beam to be degraded by aberrations and by misalignments and distortions of the focusing elements. In charged-particle optics there is the additional complication of self-fields.

An alternative point of view is to regard the emittance as representing the effect of a force tending to disperse the beam. The envelope equation (3) can equally well be expressed in terms of time as the independent variable. Multiplying Eq. 3 by term by $\gamma m_0 \dot{z}^2$ converts it into $\gamma m_0 \ddot{x}$. If we now consider the force on a small volume element of the beam rather than on individual particles as before, the second term represents the attractive focusing force, and the fourth the space charge repulsion. It is not difficult to show that the emittance term arises from a negative radial pressure gradient[10]. For a matched gaussian beam the temperature is everywhere constant, but the density and hence nkT decreases with radius. For a beam with a K-V distribution on the other hand the density is constant but the pressure decreases with radius.

Perhaps the most fundamental viewpoint, however, is obtainable from statistical mechanics. The particles in the beam represent an ensemble in phase-space evolving in time under the constraints of Liouville's theorem. As observed earlier, the fundamental constraint is that the phase-space density in the neighbourhood of a particular point remains constant. For a given distribution it is by no means immediately evident how to use this fact to determine how the various moments of the distribution, such as those specifying the r.m.s. emittance or beam radius, evolve with time. Energy must also be conserved, and it is now believed that in an external potential independent of z the <u>transverse</u> energy (kinetic + potential + field energy) observed in the beam frame is conserved. This enables correlations between radial density distribution and emittance to be made, but does not allow predictions of how either of these quantities varies with z.

At this point we make connection with the optical viewpoint and note that aberrations and non-linearities in a focusing channel, act on a beam that is not matched, (so that the distribution is not independent of z), to cause phase mixing of the osillations of individual particles, and filamentation of the phase space. Filamentation can also occur in a drifting expanding beam. Such filamentation in general increases the projected emittances of the beam. (This is not always so, artificial singular distributions can be constructed in which $\bar{\varepsilon}$ decreases monotonically[11]). Coupling, even in linear systems can cause 'twisting' of the phase-space distribution in which the values of the projected emittances ε_x and ε_y oscillate. This is observed in the well known n = 0.2 coupling resonance in cyclotrons.

EMITTANCE AND ENTROPY

A connection between emittance and entropy was first noted as long as 30 years ago. This idea was later explored in a short paper, and an attempt was made to relate the entropy to other variables regarding the beam as a 'drifting gas' described in terms of thermodynamic variables such as pressure, temperature, and internal energy[12]. This did not lead to any new insights of practical value, and the connection has perhaps acquired the status of an intellectual curiosity. It is worth re-examining the question, to see whether any useful results might be obtained. For simplicity, we start as in ref. 12 with a one dimensional system described by an ensemble of N points on the xx' plane. This plane is then divided into cells of area A, such that each cell contains many points, but the cell size is small compared with that of the overall distribution. With these assumptions, and Boltzmann's definition of entropy,

$$S = k \ln W, \tag{5}$$

where W is the number of ways that the points can be assigned to the cells to produce the given distribution, it is shown in ref. 12 that for a uniformly filled ellipse of area $\pi\varepsilon$

$$S_0 = \ln \pi\bar{\varepsilon} - \ln A, \tag{6}$$

where S_0 has been written for S/kN. (Note that $\bar{\varepsilon}$ is the r.m.s. emittance including the factor 16 in Eq. 3). For other distributions there is a different numerical factor under the logarithm in the first term on the r.h.s. of Eq. 6, leading to an additional constant to be added to S_0 in Eq. 6.

One feature that this model shows is the growth of effective emittance arising from filamentation. If the density distribution varies by only a small amount from one cell to the next, distortion of the shape of the distribution does not affect the entropy. If, however, the distribution becomes excessively filamented this is no longer the case. When the width of the filaments becomes comparable or less than the cell size more cells are occupied, and the entropy increases. If the cell size is set equal to that which can be resolved by the measuring apparatus, then the emittance increase becomes analogous to the entropy increase of two almost identical fluids that are mixed. This is the classic 'mixing' problem. If some red fluid is dropped in to an otherwise identical colourless fluid, the boundary between the two can at first clearly be seen; later mixing occurs until the whole gradually assumes a uniform pink colour.

Following the analogy between emittance and entropy, both would at first sight appear to be a measure of the disorder of the system. Unfortunately the correspondence is not so good as might first appear. For example, returning to the one dimensional system described above, if instead of the points being uniformly distributed within an elliptical contour they are distributed round its circumference only, (corresponding to particles all oscillating with the same amplitude but uniformly distributed in phase,) the entropy and emittance are very different. Because of the very small number of cells occupied the entropy will be extremely small. The r.m.s. emittance will, on the other hand, be four times as large. Likewise, for a two dimensional system the K-V distribution is represented in phase-space by a three-dimensional shell and thus has zero four-dimensional hyper-volume. This represents a highly ordered system with low entropy. The less ordered 'waterbag', in which the centre of the hyperellipsoidal shell is uniformly filled, has very much higher entropy. If, on the other hand, projections of these two distributions on the xx' plane are considered, the ellipse is filled, uniformly for the K-V distribution and with parabolic density profile decreasing to zero for the waterbag; the emittance of the less ordered waterbag distribution is now <u>lower</u>.

A further example to illustrate this is the projected emittance of a uniform beam of radius a_0 with axial symmetry confined by electrostatic lenses, but rotating with angular velocity $\dot{\theta}$ about the axis. The flow is laminar and therefore highly ordered, the temperature is zero, but the projected emittance on the xx' plane is an ellipse of area/π given by

$$\varepsilon = a_0^2\, \dot{\theta}/v_2. \tag{7}$$

In xx' yy' space the points corresponding to this distribution lie on a two-dimensional surface with entropy even less than that of a K-V distribution with the same projected emittance. (Note, however, that for a rotating beam confined by a solenoid, the emittance must be measured in the

Larmor frame. It may then be zero if $\dot{\theta}$ is zero in this frame, as would be the case, for example, with Brillouin flow).

We conclude that emittance cannot be directly related to disorder. Despite these anomalies, the correspondence between emittance and entropy might not be so bad for less singular distributions. In many practical situations the transverse velocity distribution is not far from Maxwellian. Consideration of the entropy might enable the final equilibrium configuration of a long beam to be determined, though it must first be established that such an equilibrium exists. To avoid this difficulty consideration of the following simpler problem is proposed. A parabolic potential well with axial symmetry is provided over a finite length. This might be a section of a charged cylinder which is transparent to charged particles. At time $t = 0$ a gas of charged particles with axial symmetry, radial velocity, and arbitrary radial distribution which is independent of z is released in the potential well. The particles move radially, and because of the non-linearity of the space-charge force with radius, individual particle oscillations of different amplitude will phase-mix. This is similar to, but simpler than, the problem of a mismatched beam launched into a uniform focusing channel. (If the presence of ends is objected to, a ring configuration can be substituted). Now it is reasonable to suppose that this simpler artificial system will arrive at some final equilibrium. The following questions might be asked about its behaviour.

(1) Does the particle motion remain both independent of z and without z-velocity, or does the emergence of chaotic behaviour give rise to z-variation and z-velocities?

(2) If so, does this imply that in a beam there is no final steady state with $v_z >> v_x, v_y$?

(3) Does the distribution become Maxwellian, and if so is there equipartition between radial and z directions?

In the above questions collisions have been ignored, but we know that they will ultimately lead to a Maxwellian distribution in three dimensions, with conserved energy and maximum entropy. We may further enquire:

(4) Can anything be said about relative time constants for relaxation to a Maxwellian distribution by collisions and the corresponding relaxation time by chaotic effects (if there is one)?

(5) Is the final distribution obtained by solving Poisson's equation with the Boltzmann equation[13], with energy conservation and the

constraint that the entropy should be a maximum? (The self-fields will contribute to the internal energy U of the system).

At this point it may be asked what the relevance of considering the above questions is to the problem of how the emittance and radial distribution of a beam develops. It can only give some indication of what the behaviour after a very long time might be, and this is probably of no practical interest at the present time. Transport systems of such great length do not exist, and if they did then the beam behaviour would probably be more influenced by misalignments of the focusing elements.

More generally, it may be asked whether the concepts of statistical mechanics and thermodynamics can yield new insights, or put on a more firm footing recent work on emittance growth.

CONCLUSION

Understanding of emittance growth and equipartition phenomena in particle beams where space-charge forces are large are incomplete. It is not clear whether any new physical principles remain to be discovered that might provide insights of practical value. It is hoped that the informal discussion in the present paper might suggest lines for further thought and enquiry.

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