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OPTICAL AND HYDRODYNAMICAL APPROACHES TO CHARGED PARTICLE BEAMS

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Abstract—The optical paraxial ray equation for charged particles, including the effects of self fields, is discussed in both the laboratory and rotating Larmor frames. The corresponding beam envelope equation is derived, in the first instance for laminar flow but later for some non-laminar distributions. The equivalence between optical and hydrodynamical viewpoints is discussed in some detail, and the relation between pressure and emittance is explored. Finally, some of the characteristics of longitudinal energy spread in the beam are investigated.

1. INTRODUCTION

CHARGED particle beams may be looked at from two points of view. On the one hand there is the optical approach, in which the beam is considered as a bundle of orbits, whose properties obey optical laws, derived from the principles of classical mechanics. Alternatively, the beam may be considered as a hot gas, confined by external focusing fields, for which a hydrodynamic description is appropriate.

The relation between these two approaches, and in particular the connection between the optical emittance, and hydrodynamic pressure and temperature is not immediately evident. In this paper a few simple beams are explored from both points of view, and the equivalence of the descriptions is illustrated. The aim is to provide a simple discussion of essential physical features rather than a formal and complete analysis.

Precisely what constitutes a 'beam' is not easy to define. The term is generally taken to refer to a roughly cylindrical collection of charges, in which all the members of one species have a component of velocity of the same sign parallel to the axis. Sometimes only one type of particle is present, though often the system is partly or fully 'neutralized' by approximately stationary particles of the opposite sign. In many practical beams the velocity component parallel to the axis greatly exceeds the transverse velocity, and that the energy spread of the moving particles is small, a few percent or less.

One of the commonest types of beam, that in a cathode ray tube, has extremely small energy spread and transverse velocities, and there is very little space charge interaction between the particles. At the other extreme, represented by pulsed relativistic electron beams carrying tens of thousands of amps, the behaviour is dominated by self-field and image effects, and substantial transverse velocities can develop. Collisions may also be of importance.

In the present paper we choose examples in which transverse velocities are small, but self fields may be important. Collisions are neglected unless the contrary is specifically stated.

2. PARAXIAL RAY EQUATION FOR A SINGLE CHARGE

The basic equation of linear particle optics is the paraxial ray equation. This defines the trajectory of a charge. The form that we shall use is that appropriate to a system with axial symmetry, containing longitudinal and radial components of electric and magnetic field. For the moment we assume that these are produced by external devices such as solenoids and axially symmetrical electrodes with holes for the beam to pass through. We restrict attention to trajectories near the axis, so that transverse velocities are small, and longitudinal field components are essentially independent of r, whereas radial components are proportional to r. Relativistic units are used, $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$, from which it follows that the particle momentum is $p = \beta\gamma m_0 c$ and the kinetic energy is $(\gamma - 1)m_0c^2$. Differentation with respect to z is denoted by primes, so that $\mathrm{d}r/\mathrm{d}t = \beta c r'$. Because of the paraxial approximation, in which angles with the axis are small, the momentum in the z-direction is essentially independent of the transverse momentum, $p_z \approx p$. The E_z field may be written as $\gamma' m_0 c^2/q$, and the magnetic field B_z as $-2\gamma m_0 \Omega_L/q$, where Ω_L is the Larmor frequency, equal to half the cyclotron frequency, $\Omega_L = \omega_c/2$. (The negative sign indicates anticlockwise rotation in a positive field.)

In these units, the paraxial equation may be written

$$r'' + \frac{\gamma' r'}{\beta^2\gamma} + \left(\frac{\gamma''}{2\beta^2\gamma} + \frac{\Omega_L{}^2}{\beta^2c^2}\right)r - \left(\frac{P_\theta}{\beta\gamma m_0 c}\right)^2\frac{1}{r^3} = 0. \qquad (1)$$

The symbols have been defined above except for P_θ, which represents the canonical angular momentum of the charge about the axis, $P_\theta = \gamma m_0 r^2\dot{\theta} + qA_\theta r$.

This equation, and its derivation, may be found in most texts on electron optics; the notation varies however and superficially it may present a different appearance. The form quoted here and its derivation, using the principle of least action, may be found in the text of PANOFSKY and PHILLIPS (1955). More directly the equation may be derived by finding the equation of motion of a particle in fields of the appropriate form, either directly or from the appropriate Lagrangian, and then changing the independent variable from t to z. PIERCE (1954) in a non-relativistic formulation uses a direct physical approach; KIRSTEIN, KINO and WATERS (1967) start from the non-relativistic Lagrangian.

The equation contains only the z-components of the fields on the axis, radial components do not occur explicitly. This is because the divergence of the fields is zero, so that there is a relation between the field gradients along the axis and the radial fields. Thus the term containing γ'' is proportional to the radial electric field.

The given axial fields determine γ' and Ω_L, so that for a particle of given γ at $z = 0$, the value of r as a function of z is determined for initial conditions r_1, r_1' and for P_θ which is a constant of the motion.

The angular co-ordinate is determined from the subsidiary equation

$$\theta - \theta_1 = \frac{1}{\beta c}\int_0^z \left(\Omega_L + \frac{P_\theta}{\gamma m_0 r^2}\right)\mathrm{d}z. \qquad (2)$$

That this is so may be seen by writing

$$P_\theta = p_\theta + qA_\theta r = p_\theta + \tfrac{1}{2}qB_z r^2 = p_\theta - \gamma m_0 \Omega_L r^2, \qquad (3)$$

so that the integrand becomes $p_\theta/\gamma m_0 r^2 = \omega$.

We now look at some very simple solutions of equations (1) and (2), in which the electric field is zero. With this restriction equation (1) simplifies to

$$r'' + \frac{\Omega_L{}^2}{\beta^2 c^2} r - \left(\frac{P_\theta}{\beta\gamma m_0 c}\right)^2 \frac{1}{r^3} = 0. \tag{4}$$

If further, the magnetic field is uniform, we know that the solution represents a helical trajectory, of pitch $2\pi\beta c/\omega = \pi\beta c/\Omega_L$.

If $P_\theta = 0$, then the helix passes through the axis; the solution is $r = r_1 \sin(\Omega_L z/\beta c)$; one period in z represents two revolutions of the particle. A particle moving from a zero field into a field of B_z is illustrated in Fig. 1a. The radial field components at the end of the solenoid impart the angular velocity about the axis.

As a second example we make r constant in a uniform magnetic field. There are obviously two such solutions, one representing a helix centred on the axis, and the other a particle moving in a straight line along a field line. To find the appropriate value of P_θ we set the first term in equation (4) equal to zero. This yields

$$P_\theta = \pm\gamma m_0 \Omega_L r^2. \tag{5}$$

Figure 1b shows the trajectory of an electron passing from a field B_z, where Ω_L is $-qB_z/2\gamma m_0$, to an equal field of opposite sign. This illustrates both types of solution; the azimuthal impulse from the radial fields associated with the cusp geometry is twice as great as that in the first example, so that the radius of the helix is twice as large.

If B_z varies slowly with z, so that the r'' term in equation (1) is small, equation (5) still holds; since P_θ is a constant of the motion, $\gamma r^2 \Omega_L$, or $r^2 B$, is invariant as expected.

We note now a useful transformation which simplifies the equation. If the system is viewed in a frame of reference rotating with frequency ω_f, then axial and radial

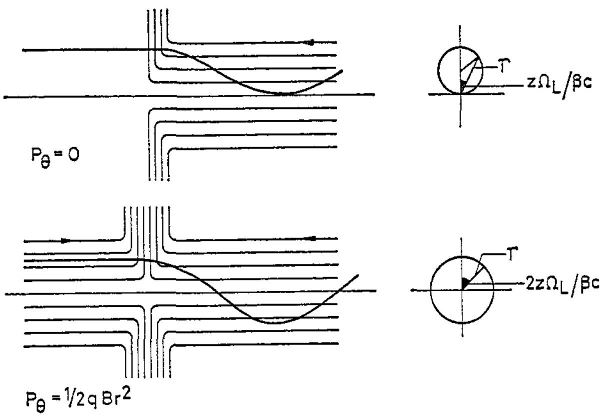


FIG. 1.—Two solutions of the paraxial ray equation (4), are represented. In the first of these p_θ is zero and Ω_L jumps from zero to a constant value. The trajectory passes through the axis, and rotates about it with frequency Ω_L. In the second, Ω_L changes sign at the transition point, and a trajectory initially parallel to the axis rotates at constant radius with frequency $2\Omega_L$. In the Larmor frame, the particle in the first diagram oscillates in the magnetic field region in a plane through the axis from one side of the beam to the other. The particle in the second diagram spirals round the axis with angular velocity Ω_L on both sides of the transition region.

components of field are transformed according to the laws calculated in the Appendix. In particular, if only a field B_z is present, and $\omega_f = \Omega_L$

$$E_{r2} = -\tfrac{1}{2}r\Omega_L B_{z1} = -\gamma m_0 \Omega_L{}^2 r/q \qquad (6)$$
$$B_{z2} = 0,$$

where suffices 1 and 2 refer to the original and rotating frames. The force is now purely radial, and equations (4) and (2) become

$$r'' + \left(\frac{\Omega_L}{\beta c}\right)^2 r - \left(\frac{p_\theta}{\beta\gamma m_0 c}\right)^2 \frac{1}{r^3} = 0 \qquad (7)$$

$$\theta - \theta_1 = \frac{1}{\beta c}\int_0^z \frac{p_\theta \, \mathrm{d}z}{\gamma m_0 r^2}, \qquad (8)$$

where p_θ is now the *mechanical* angular momentum in the rotating frame. This frame is called the "Larmor frame". The trajectory shown in the right hand side of Fig. 1a becomes sinusoidal and also planar when viewed in this frame. The trajectory in Fig. 1b becomes a spiral of pitch $2\pi\beta c/\Omega_L$.

Since in the Larmor frame the force is towards the axis, the projections of all trajectories on two orthogonal planes Ox and Oy through the axis are sinusoidal. Equations (7) and (8) may therefore be replaced by the simpler linear equations

$$\begin{aligned} x_L'' + \left(\frac{\Omega_L}{\beta c}\right)^2 x_L &= 0 \\ y_L'' + \left(\frac{\Omega_L}{\beta c}\right)^2 y_L &= 0, \end{aligned} \qquad (9)$$

where the suffix L denotes that x_L and y_L are projections in the rotating Larmor frame.

Initial conditions may be expressed in terms of x_L, x_L', y_L, y_L' rather than r, r', θ, P_θ. If P_θ is zero, the orbits pass through the axis, and this implies that x and y are simultaneously zero.

Although the discussion has been confined to equation (5), the same observations apply to equation (1). The forces arising from the electric field are not velocity dependent, and remain unchanged during the transformation to the Larmor frame. Equation (1) in the Larmor frame is therefore unchanged except that P_θ is replaced by p_θ. The equation of the x projection is

$$x_L'' + \frac{\gamma' x'_L}{\beta^2\gamma} + \left(\frac{\gamma''}{2\beta^2\gamma} + \frac{\Omega_L{}^2}{\beta^2 c^2}\right) x_L = 0. \qquad (10)$$

In the absence of electric fields in the laboratory frame it takes the simple form

$$x_L'' + x_L/\lambda\!\!\!^{-2}(z) = 0 \qquad (11)$$

where $\lambda\!\!\!^{-}(z) = \beta c/\Omega_L(z)$.

In the presence of electric fields, equation (10) can be simplified by a transformation to remove the x_L' term. This transformation, to "reduced variables" $X_L = x_L(\beta\gamma)^{1/2}$ yields after a little manipulation

$$X_L'' + \left(\frac{\gamma'^2(\gamma^2+2)}{4\beta^4\gamma^4} + \frac{\Omega_L{}^2}{\beta^2 c^2}\right) X_L = 0, \qquad (12)$$

which is of the same form as equation (11). This transformation is equivalent to 'Picht's transformation' $X = V^{1/4}x$ of non-relativistic electron optics. We shall use it in Section 8.

Having reduced the paraxial equation to these simple forms, we now consider the effects of self-fields arising from currents and charges in the beam.

3. SELF-FIELDS IN A CYLINDRICAL BEAM

In order to preserve linearity, and thereby the simplicity of our discussion, we consider the self-fields in a beam which has uniform charge and current density, moving through a stationary background of ions with charge density equal to a fraction f of that of the beam particles. Associated with the charge and current of the beam are a radial electric field and azimuthal magnetic field, both proportional to radius. The electric field acts directly on a charge in the beam to produce an outward force; the magnetic field acts also on the moving charges to produce a $v \times B$ force, which acts radially inward. The net force is the sum of these two. The values can readily be calculated from Gauss' theorem and Ampere's law respectively. If N is the number of particles per unit length, a the beam radius, and r the co-ordinate of a charge, then

$$E_r = \frac{N(1-f)qr}{2\pi\varepsilon_0 a^2}, \qquad B_\theta = \frac{Nqr\mu_0\beta c}{2\pi a^2}. \tag{13}$$

These give rise to forces in the radial direction

$$F_E = \frac{Nq^2r}{2\pi\varepsilon_0 a^2}(1-f), \qquad F_M = -\frac{Nq^2rc^2\mu_0\beta^2}{2\pi a^2} \tag{14}$$

Writing $\mu_0\varepsilon_0 = 1/c^2$, the total force is

$$F = \frac{Nq^2r}{2\pi\varepsilon_0 a^2}(1-f-\beta^2). \tag{15}$$

The acceleration arising from this force is $\ddot{r} = F/\gamma m_0$, whence $r'' = F/\gamma m_0\beta^2c^2$. Setting $q^2/4\pi\varepsilon_0 m_0c^2 = r_0$, the classical radius of the charge, yields

$$r'' = \frac{2Nr_0r}{\beta^2\gamma a^2}(1-\beta^2-f). \tag{16}$$

Writing $Nr_0 = \nu$

$$K = \frac{2\nu}{\beta^2\gamma}(1-\beta^2-f), \tag{17}$$

it is now possible to add a term $-Kr/a^2(z)$ to the left hand side of equation (1) to take account of the self fields. In order to find $r(z)$ from this equation we need to know $a(z)$. This itself is a function of the collective properties of the orbits; the relation between them, essential for self-consistency, is studied later. For the moment we assume all the orbits to be geometrically similar, so that for any particular particle, $r \propto a$. For a particle at the edge of the beam, $r = a$.

The quantity K determines the effect of the self fields on the particle orbits; it may be termed 'generalized perveance'. For $f = 0$, and non-relativistic beams,

$K = 2Nr_0/\beta^2$, which is proportional to $I/V^{3/2}$, the perveance as normally defined. For completely neutral beams, $K = -2\nu/\gamma = -2I/I_A$ where I_A is the Alfvén current $4\pi\varepsilon_0\beta\gamma m_0c^3/q$, for electrons equal to 17,000 $\beta\gamma$ amps. (ALFVÉN, 1939). If $f = 1 - \beta^2 = 1/\gamma^2$ then $K = 0$ and the electric and magnetic forces balance; this is sometimes known as the 'Budker condition'. (BUDKER, 1956).

In the absence of external focusing, only the first term of equation (1) remains. In the presence of the perveance term Kr/a^2, the trajectory equation of a charge at the edge of the beam, where $r' = a$, is

$$rr'' = K. \tag{18}$$

If all trajectories are parallel to the axis where $r = 0$, then for positive K this defines the well known 'beam spreading' curve; for negative K the solution represents a pinch. (LAWSON, 1958).

Another interesting example is provided by retaining a uniform magnetic field and setting $r'' = 0$; this represents a beam in which the outward space charge and centrifugal forces on every particle are exactly balanced by the Lorentz force. For $P_\theta = 0$ this is known as 'Brillouin flow'. (BRILLOUIN, 1945). Only the term $\Omega_L{}^2r/\beta^2c^2$ remains from equation (1). Setting this equal to the perveance term, with $r = a$, yields

$$K/a^2 = \Omega_L{}^2/\beta^2c^2. \tag{19}$$

For an un-neutralized non-relativistic beam, from equation (17)

$$K = 2Nq^2/4\pi\varepsilon_0m_0\beta^2c^2 = a^2\omega_p{}^2/2\beta^2c^2, \tag{20}$$

where ω_p is the plasma frequency, so that equation (19) becomes

$$\omega_p{}^2 = 2\Omega_L{}^2 = \tfrac{1}{2}\omega_c{}^2. \tag{21}$$

In the laboratory frame it represents that particular balance between centrifugal, space charge and Lorentz forces, for which $P_\theta = 0$. In the Larmor frame on the other hand the balance is between radial electric field and space charge forces only. The trajectories are straight lines parallel to the axis.

4. PARTICLE ACCELERATOR BEAMS

Focusing by magnets in cyclic particle accelerators is also described by equations of the form of equation (11), but with x in place of x_L. The axis however is in general curved, and the values of $\lambda\!\!\!^{-2}(z)$ are different in the orbit plane, and a direction perpendicular to this plane. Indeed, in strong focusing systems $\lambda\!\!\!^{-2}$ is of opposite sign in the two directions, and in both it alternates in sign as z varies. In particular, in a quadrupole focusing magnet, $\lambda\!\!\!^{-2}$ has the same magnitude but opposite signs in the two planes.

As in the cylindrical paraxial equation, the self-fields provide an extra linear focusing (or defocusing) term. The beam is no longer circular however, but elliptical, and this introduces the complication that the focusing in the x direction is a function of the beam dimension b in the y direction. If fields arising from image forces in the walls are neglected, then the equations corresponding to equation (11).

but including the self-field terms are

$$x'' + \left(\frac{1}{\lambda_x^2} - \frac{2K}{a(a+b)}\right)x = 0$$
$$y'' + \left(\frac{1}{\lambda_y^2} - \frac{2K}{b(a+b)}\right)y = 0. \tag{22}$$

For the special case of $\lambda_x = \lambda_y$, $a = b$, these reduce to

$$x'' + \left(\frac{1}{\lambda^2} - \frac{K}{a^2}\right)x = 0. \tag{23}$$

Since the purpose of the present paper is pedagogical, we confine attention henceforth to this particular case, but note that generalization to an elliptical beam is straightforward provided that there is no longitudinal magnetic field B_z. If there is such a field the behaviour is complicated; fortunately such a configuration is not often encountered.

Practical accelerator beams tend to have density profiles of gaussian shape, but for many purposes the uniform density model is a good approximation.

5. ENVELOPE EQUATION FOR LAMINAR BEAM

The motion of a particular electron in a beam is described by equation (1) with the additional perveance term $-Kr/a^2$. For a beam in which all the orbits are geometrically similar, the envelope equation is obtained by putting $x = a$ in equation (1) and including the perveance term; this then becomes

$$a'' + \frac{\gamma' a'}{\beta^2\gamma} + \left(\frac{\gamma''}{2\beta^2\gamma} + \frac{\Omega_L{}^2}{\beta^2 c^2}\right)a - \left(\frac{P_{\theta a}}{\beta\gamma m_0 c}\right)^2 \frac{1}{a^3} - \frac{K}{a} = 0, \tag{24}$$

where $P_{\theta a}$ refers to a charge at the edge of the beam. In paraxial approximation the contribution to the B_z field arising from the current associated with P_θ may readily be shown to be negligible.

In the absence of external electric fields, γ' and γ'' are zero and equation (24) simplifies to

$$a'' + \frac{a}{\lambda^2(z)} - \frac{K}{a} - \left(\frac{P_{\theta a}}{\beta\gamma m_0 c}\right)^2 \frac{1}{a^3} = 0. \tag{25}$$

In the Larmor frame the equation is identical, except for $p_{\theta a}$ in place of $P_{\theta a}$.

The physical significance of this equation is straightforward, it represents a balance between the inertial force (first term) and the focusing, self and centrifugal forces respectively.

Some special solutions are of interest; two of these, corresponding to terms 1 and 3 only (equation 18) and terms 2 and 3 only (equation 19) have already been discussed. The first and last terms only, represent the beam in free space; a manifold of straight lines produces the hyperboloidal envelope

$$a^2 = a_0^2 + \left(\frac{P_{\theta a}}{a_0\beta\gamma m_0 c}\right)^2 z^2. \tag{26}$$

The last two terms only of equation (25) represent a beam in which the self force balances the centrifugal force. This can only happen when K is negative so that the inward magnetic force dominates. For a completely neutralized beam $K = -2\nu/\gamma$; writing $p_{\theta a} = \beta_\perp \gamma m_0 c$ the last term becomes $(\beta_\perp/\beta)^2$, and the equilibrium condition is the familiar relation

$$\left(\frac{\beta_{\perp a}}{\beta}\right)^2 = \frac{2\langle \beta_\perp{}^2 \rangle}{\beta^2} = \frac{2\nu}{\gamma}. \tag{27}$$

Sketches of the solutions corresponding to the various combinations of terms in equation (25), with λ independent of z, are shown in Fig. 2. Some, but not all, of these have been already discussed.

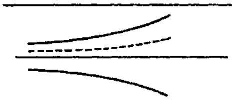
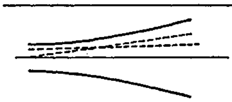
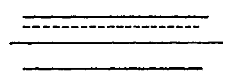
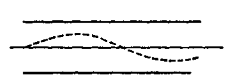
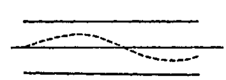
Envelope, and orbits in Larmor frame (dotted)	Terms in equation 25	Equation in text
	1,2	–
	1,3 (K positive)	18
	1,3 (K negative)	18
	1,4	26
	2,3	19
	2,4	–
	3,4	27

FIG. 2.—These curves represent solutions of the envelope equation (25) in which only two of the four terms are present. The dotted lines represent projections of typical trajectories in the Larmor frame.

In the discussion so far we have been concerned only with laminar beams. There is no distinction between an orbit model and a hydrodynamic model, the force per unit volume is just the number density times the force on a single particle. The beam is cold, and exerts no pressure.

Beams with finite pressure and temperature will now be described.

6. AN 'AZIMUTHALLY HOT' BEAM

The analysis in previous sections has been confined entirely to laminar beams. The paraxial equation however also applies to beams in which some or all of the particles have canonical angular momenta equal to $-P_\theta$, this is evident because only the square of P_θ occurs in the equation.

As a simple example, we consider a beam in the Larmor frame, in which equal numbers of particles have angular momentum p_θ and $-p_\theta$.

The paraxial equation in this context is best interpreted in a hydrodynamical sense. Although the equation is the same as that for which all particles have the same sign for p_θ, the physical interpretation is different. Instead of a rotating volume element with zero pressure, there is a stationary volume element, with finite pressure. Instead of a centrifugal force, there is a force arising from a pressure gradient. To demonstrate explicitly that the term has the same form, we note that the velocity v_θ a particle at radius r is $\pm p_\theta(r)/\gamma m_0 r$. Writing Π rather than p for the pressure to avoid confusion, the element $\Pi_{\theta\theta}$ of the pressure tensor divided by the number of electrons n_0 per unit volume is

$$\frac{\Pi_{\theta\theta}}{n_0} = \frac{\gamma m_0}{n_0}\int (v_\theta - \bar{v}_\theta)^2 n(r, \theta, v_\theta)\, dv_\theta = \frac{{p_\theta}^2}{\gamma m_0 r^2}. \tag{28}$$

The outward force on this element arising from the pressure gradient is minus the divergence of the pressure tensor, which, in the absence of radial and longitudinal temperature may be shown to be $-\Pi_{\theta\theta}/r$. The force per electron is therefore just $p_\theta^2/\gamma m_0 r^3$ as expected. The last term of the paraxial equation (1) could equally well have been written

$$\left(\frac{p_\theta}{\beta\gamma m_0 c}\right)^2 \frac{1}{r^3} = \frac{F_r}{\gamma m_0 \beta^2 c^2} = \frac{1}{\gamma m_0 \beta^2 c^2 n_0}\frac{\partial \Pi_{\theta\theta}}{\partial r} \tag{29}$$

The argument so far has referred to the Larmor frame. It is readily verified that in the laboratory frame the fluid velocity at a point in this model is equal to the Larmor frequency times the radius, and that the centrifugal force on a volume element balances the $j_\theta \times B_z$ force.

The temperature of the beam in a given direction may be defined by the relation $n_0 kT = \Pi_{ii}$; in the azimuthal direction the temperature in either frame is accordingly $\Pi_{\theta\theta}/n_0 k$, which is given by equation (28).

From equation (29), $\Pi_{\theta\theta} \propto n_0 r^2$ for constant γ and p_θ. Regarding the beam as a two dimensional gas of volume $V \propto a^2$, this implies that ΠV^2 is constant. The gas obeys the adiabatic law $\Pi V^\gamma =$ constant where γ is the adiabatic exponent, equal to $(2 + 2)/2 = 2$. (Although illustrated here in a special context, the adiabatic gas laws for a collisionless system may easily be found directly from Liouville's theorem).

7. A BEAM WITH ISOTROPIC TRANSVERSE TEMPERATURE

Although a natural form to assume for the transverse velocity distribution in a finite temperature beam might be Maxwellian, this is not consistent with the uniform density which has been assumed so far. We have already studied a very simple finite temperature distribution associated with a beam of uniform density; we now introduce another, which has an isotropic, though radius dependent, pressure. This is the 'microcanonical' distribution of KAPCHINSKIJ and VLADIMIRSKIJ (1959), which represents the paraxial limit of the distribution more recently studied by HAMMER and ROSTOKER (1970). The name reflects the fact that, in a beam which is uniform in the z direction, so that a is constant, all particles have the same transverse energy. The distribution is

$$\frac{x^2+y^2}{a^2}+\frac{x'^2+y'^2}{\alpha^2}-1=0. \tag{30}$$

This distribution function represents a three dimensional hyper-ellipsoidal shell in four dimensional $xx'yy'$ space. It may be shown, by a generalization of the theorem of Archimedes, that a projection of such a shell on a plane is an elliptical area of uniform density. In particular, the density in the xy plane (perpendicular to the beam axis) is uniform and bounded by a circle of radius a. In the xx' or yy' plane the projection is an ellipse with semi-axes a and α. The angle α represents the maximum angle a trajectory makes with the axis.

The projections of individual orbits in the xx' or yy' planes consist of concentric ellipses, representing simple harmonic motion in the uniform focusing field. In the xy plane however the projections are ellipses with semi-axes A and B such that $A^2+B^2=a^2$. The axes are distributed uniformly with θ, (since there is no preferred direction); when $A=B$ they are circles with radius $a\sqrt{2}$; when $B=0$ they are straight lines of length $2A=2a$, the orbits under these conditions extending to the edge of the beam, as illustrated in Fig. 3.

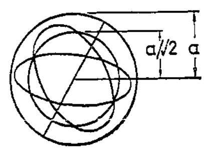


FIG. 3.—Projections on a plane perpendicular to the axis of some typical orbits in a uniform diameter beam with the microcanonical distribution defined by equation (30). The velocity of particles passing through any point on the projection is constant and isotropic, decreasing parabolically from a maximum value for the central point to zero for a point at the edge of the beam.

From equation (30), it is evident that at any radius $(x^2+y^2)^{1/2}$, the value of $x'^2+y'^2$ is constant. Furthermore, because of the symmetry between x' and y', and the fact that there is no preferred orientation of the axes, it follows that in the xy plane the angular distribution of trajectories passing through any point is uniform. Setting $x'^2+y'^2=v_\perp{}^2/\beta^2c^2$, equation (30) may be written

$$v^2=\beta^2c^2\alpha^2(a^2-r^2)/a^2. \tag{31}$$

Since the distribution at a point is isotropic, $\pi_{rr}=\pi_{\theta\theta}$, the transverse pressure is a scalar,

$$\Pi=\tfrac{1}{2}n_0\gamma m_0\beta^2c^2\alpha^2(a^2-r^2)/a^2. \tag{32}$$

Unlike the previous example, the pressure is a maximum on the axis and decreases to zero at the beam edge. From equation (32) the force per unit volume may be found,

$$\frac{\partial\Pi}{\partial r}=n_0\gamma m_0\beta^2c^2\alpha^2r/a^2. \tag{33}$$

Normalized to a single particle this becomes, at $r = a$,

$$F_r = \gamma m_0 \beta^2 c^2 \alpha^2 / a = \gamma m_0 \beta^2 c^2 \varepsilon^2 / a^3 \tag{34}$$

where we have written

$$\varepsilon = a\alpha. \tag{35}$$

The quantity ε is defined as the 'emittance' of the beam. It is seen to be $1/\pi$ times the area occupied by the trajectories projected on to the xx' or yy' plane. In our example the x and y axes, and the x' and y' axes of the hyperellipsoid (equation 30) were taken as equal; in general they need not be, and the emittance in the two planes can be different. This is the usual situation in particle accelerators, where the horizontal and vertical focusing are not of equal strength.

Comparison of equation (34) with equation (29) shows that the force on a volume element associated with the emittance ε is of the same form as that associated with the finite angular momentum in a laminar beam. Comparing equations (25) and (34)

$$\varepsilon = a\alpha = \frac{p_{\theta a}}{\beta \gamma m_0 c} = \frac{a\beta_{\theta a}}{\beta} = a\theta_a, \tag{36}$$

where θ_a is the helix angle at the edge of the beam.

It might be conjectured at this point that the envelope equation for a beam with emittance ε would be as equation (25), but with the last term ε^2/a^3. To make this statement precise, the meaning of ε must be generalized, since it has so far been defined only in the context of a beam with $a' = 0$. This may be done from equation (32). Assuming again that the gas obeys the law $\Pi V^2 = \text{const}$, where Π is measured at $r = 0$, it follows that the r.h.s with $r = 0$ multiplied by a^4 must be constant. Since $n_0 \propto a^{-1}$, then α must be $\propto a^{-2}$, so that $\varepsilon = \alpha a$ is invariant. In the next section we demonstrate from a different point of view that this area is invariant, though the shape and orientation of the projected ellipse vary as we move away from a region where $a' = 0$. The distribution function corresponds to a hyperellipsoid with axes no longer parallel to the co-ordinate axes; superimposed on the isotropic velocity distribution at a point in the beam is a radial velocity proportional to r, but the pressure is still given by equation (33).

From equation (36) it is evident that in the terms describing the angular momentum of a laminar beam, or the emittance of a microcanonical beam, α and θ_a are equivalent. Indeed, the projections on the xx' plane of the orbits in both distributions are identical. As already stated, the microcanonical distribution represents a three dimensional shell in $xx'\ yy'$ space, it may readily be verified that the distribution in the laminar beam represents a two dimensional shell, defined by the intersection of the three dimensional volumes

$$x = y'a^2/\varepsilon \qquad y = -x'a^2/\varepsilon. \tag{37}$$

It is left to the reader to interpret the appropriate curves in Fig. 2 in terms of a finite emittance rather than a finite angular momentum.

In this section we have taken a beam with a particular transverse velocity distribution, and shown how the paraxial ray equation may be interpreted in a hydrodynamical sense. We now complete the discussion by writing it in completely hydrodynamical form, with time as independent variable, for a beam viewed in the Larmor frame in which we also specify that $E_z = 0$.

Since $E_z = 0$, the z velocity is constant. The radial velocity of a volume element is u_r. The equation expresses the radial acceleration of this element under the action of focusing, space-charge, centrifugal and pressure gradient forces.
The basic equation is

$$\frac{Du_r}{Dt} = \frac{1}{n_0 m}\left(\Sigma F_r - \frac{\partial \Pi_\perp}{\partial r}\right), \tag{38}$$

where $\Pi_\perp$ refers to the transverse pressure. The forces are observed in the moving frame, and consequently suitable Lorentz transformations must be applied to the parameters which determine them. Furthermore, t on the left hand side must be multiplied by γ to allow for time dilation when returning to the laboratory frame. For example, in the pressure term n_0 must be multiplied by γ, and Π divided by γ. In the perveance term the self magnetic field of the electrons vanishes, but that arising from any ions must be included. The reader will readily verify that for a rotating microcanonical distribution in the Larmor frame equation (38) becomes

$$r'' + \frac{r}{\lambda^2(z)} - \frac{Kr}{a^2} - \left\{\left(\frac{P_{\theta a}}{\beta\gamma m_0 c}\right)^2 + \varepsilon^2\right\}\frac{r}{a^4} = 0. \tag{39}$$

Setting $r = a$, this becomes the envelope equation (25), with the addition of a pressure term. Unlike previous equations, this contains the effect of rotation as well as finite pressure. It is not difficult to show that the projected distribution on the xx' plane is elliptical with area equal to $1/\pi$ times the square root of the expression in curly brackets.

8. THE EMITTANCE CONCEPT

Starting with the optical paraxial ray equation we have moved in the previous sections towards a hydrodynamic description of our beam, making use of the concepts of pressure and temperature. These have been expressed in terms of the emittance, a quantity closely associated with the transverse distribution function. So far, only the emittance of rather special distribution functions has been considered.

More generally, we can usefully apply the concept to any beam which has two planes of symmetry Ox and Oy. The emittance ε_x of the beam where $z = z_0$ can be defined as $1/\pi$ times the projected area on the xx' plane occupied by points corresponding to the trajectories as they pass the point z_0. In practical beams the distribution does not have a sharp edge. The emittance can then be specified as the minimum area enclosing a given fraction of the points, or as the area where the density of points exceeds some given fraction of the maximum density.

If we consider a group of points in phase space corresponding to particles passing the point z_0 at some instant, then by Liouville's theorem the density of these points, and consequently the volume which they occupy, is invariant. If, furthermore, the x, y and z motion is uncoupled, the area which the projection occupies on the x,p_x plane is invariant as the particles move along the z axis. In paraxial approximation x' is related to the transverse momentum as $x' = p_x/\beta\gamma m_0 c$. It follows therefore that $\beta\gamma\varepsilon$ remains constant. This quantity is sometimes called the normalized emittance. Although the area of the projection on the x, $\beta\gamma x'$ plane remains constant, the shape does not. How this varies for a linear system will be seen in the next section.

A good discussion of the emittance in an experimental context, and its relation to optical brightness is given by SEPTIER, (1967). We do not discuss this further, but now derive formally the envelope equation for a beam with finite emittance.

9. OPTICAL DERIVATION OF ENVELOPE EQUATION

We now derive the projected envelope equation and discuss its solutions in terms of the emittance concept. No further mention will be made of hydrodynamic quantities. We allow for an accelerating electric field along the beam, and use as our starting point the equation in normalized variables, equation (12). The method closely follows that of GARREN (1969). For a system with finite B_z and axial symmetry, X is measured in the Larmor frame. This analysis is also relevant to accelerators where B_z is zero, the x and y directions are uncoupled, and the focusing in the two planes is different, $(\lambda\!\!\!^{-}_x \neq \lambda\!\!\!^{-}_y)$.

We write the paraxial equation in the form

$$X'' + X/\lambda\!\!\!^{-}_X{}^2 = 0, \tag{40}$$

where λ_X is a function of z. This equation is now transformed by the introduction of phase-amplitude variables

$$X = Aw(z)\cos(\psi(z) + \Phi). \tag{41}$$

The meaning of the various quantities will be evident later, as will the significance of the seemingly arbitrary restriction on ψ imposed by the relation $\psi' = 1/w^2$. Using this relation we can write X' as

$$X' = A\left\{w'(z)\cos(\psi(z) + \Phi) - \frac{1}{w}\sin(\psi(z) + \Phi)\right\}. \tag{42}$$

Four quantities w, ψ, A and Φ have been introduced; of these w and ψ are functions of z and are the same for all trajectories, whereas A and Φ are independent of z and replace the values of x and x' at $z = 0$ as constants specifying a particular trajectory. Substituting equation (41) into equation (40) it is found that w satisfies the equation

$$w'' + \frac{w}{\lambda\!\!\!^{-}_X{}^2(X)} - \frac{1}{w^3} = 0. \tag{43}$$

This equation has many solutions, and it remains to choose an appropriate one. To do this we determine the relation between A, X, X' w and w' found by eliminating ψ from equations (41) and (42), (using the fact that $\cos^2\psi + \sin^2\psi = 1$). This may be written

$$A^2 = \frac{X^2}{w^2} + (wX' - w'X)^2, \tag{44}$$

or

$$A^2 = \gamma_0 X^2 + 2\alpha_0 XX' + \beta_0 X'^2, \tag{45}$$

where

$$\begin{aligned} \alpha_0 &= -ww' \\ \beta_0 &= w^2 \\ \gamma_0 &= w^{-2} + w'^2 = (1 + \alpha_0^2)/\beta_0. \end{aligned} \tag{46}$$

An appropriate choice of initial conditions for following the development of a beam consisting of particles corresponding to points on the ellipse represented by equation (45) is found by matching initial values of w and w' to α_0 and β_0 in accordance with equation (46). Points lying on a particular ellipse have a unique value of A, and a range of Φ from 0 to 2π. Points corresponding to trajectories with a different value of A lie on an ellipse scaled in size but otherwise geometrically similar. As z varies, the shape and orientation of the ellipse change in accordance with the variation of w and w', and points corresponding to a given value of Φ move round the ellipse. The area of the ellipse defined by equation (45) is

$$S = \pi A^2(\beta_0\gamma_0 - \alpha_0^2)^{-1/2}. \tag{47}$$

From equation (46) the quantity in brackets is equal to unity, so that the area is invariant and equal to πA^2.

Consider now an ensemble of trajectories corresponding to points uniformly distributed inside the ellipse; as z varies these all move on geometrically similar ellipses, with the appropriate value of A. Since the ellipse area is invariant, the density of points remains constant, as expected from Liouville's theorem; furthermore, since its area is just $\pi \times$ the normalized emittance of the beam, it follows that $A_0 = (\beta\gamma\varepsilon)^{1/2}$ where A_0 is the maximum value of A. The envelope equation of a beam consisting of trajectories constituting a uniformly filled ellipse in xx' space (all values of Φ, $A < A_0$) can readily be calculated. At any value of z, the edge particle in the beam corresponds to $A = A_0 = (\beta\gamma\varepsilon)^{1/2}$, $\psi + \Phi = 0$. From equation (41), $X = A_0 w = (\beta\gamma\varepsilon)^{1/2}w$ for such a particle; substituting in equation (43) it follows that the envelope equation is

$$X'' + \frac{X}{\lambda_X^2} - \frac{\beta^2\gamma^2\varepsilon^2}{X^3} = 0. \tag{48}$$

This equation is valid in the presence of an accelerating electric field. In the absence of such a field not only is it true that $X = (\beta\gamma)^{1/2}x$ but also $X'' = (\beta\gamma)^{1/2}x''$, so that equation (48) reduces to the form

$$x'' + \frac{x}{\lambda_x^2} - \frac{\varepsilon^2}{x^3} = 0. \tag{49}$$

Equations (45) and (46) specify the parameters of the emittance ellipse; evidently when $w' = 0$, at a waist in the beam, the ellipse is upright. The detailed behaviour of such ellipses has been studied in connection with beam transport systems (BANFORD 1962).

Only in linear systems, where the paraxial equation applies, does the emittance diagram remain elliptical. Aberrations produce distortion, which in extreme cases produces filamentation of the diagram and an effective increase of emittance. This phenomenon is widely discussed in the accelerator literature; a treatment in the spirit of the present paper, in which the connection between emittance and entropy is explored, is given by LAWSON, LAPOSTOLLE and GLUCKSTERN (1973).

10. LONGITUDINAL ENERGY SPREAD

So far only the effects of transverse momentum spread have been discussed; all particles have been assumed to have the same momentum along the beam. In many situations, where particles are accelerated in a gun or ion source for example, this is a good approximation. In accelerators however, especially when the beam is bunched, longitudinal momentum spread, and the corresponding emittance, are of importance.

The properties of such beams will not be discussed here, but one important feature should be noted. This is the fact that, even for non-relativistic beams, the energy spread measured in a frame moving with the beam is considerably less than that measured in the laboratory. Accelerating a beam away from a cathode by means of a grid placed close to it lowers the longitudinal temperature, but leaves the transverse temperature unchanged.

To illustrate this effect, we consider two particles in the beam with energies $\gamma m_0 c^2$ and $(\gamma + \Delta\gamma) m_0 c^2$, where $\Delta\gamma m_0 c^2$ is a typical energy difference seen in the Laboratory frame. Taking $\Delta\gamma \ll \gamma$, it follows that $\Delta\beta = \Delta\gamma(\mathrm{d}\beta/\mathrm{d}\gamma) = \Delta\gamma/\beta\gamma^3$. The difference of velocities seen in a frame moving with one of the particles, $\Delta\beta_2$, follows from the law of addition of velocities,

$$\Delta\beta_2 = \frac{\Delta\beta}{1 - \beta(\beta + \Delta\beta)} = \gamma^2 \Delta\beta = \frac{\Delta\gamma}{\beta\gamma}. \tag{50}$$

For $\Delta\gamma/\gamma$ small this is always non-relativistic, (even though $\Delta\gamma$ may be large), so that the apparent energy in the moving frame is

$$\Delta\gamma_2 m_0 c^2 = \tfrac{1}{2}(\Delta\gamma/\beta\gamma)^2 m_0 c^2. \tag{51}$$

This is always very much less than $\Delta\gamma m_0 c^2$.

An interesting example is provided by a non-relativistic beam accelerated from a thermionic cathode of temperature T_c. $\Delta\gamma$ is of the order of kT_c/m_0c^2, and for a non-relativistic beam, from equation (51)

$$\frac{\Delta\gamma_2}{\Delta\gamma} \approx \frac{T_2}{T_c} \approx \frac{\frac{1}{4}kT_c}{\frac{1}{2}\beta^2 m_0 c^2}. \tag{52}$$

The temperature in the accelerated beam is thus reduced in the ratio of the beam energy to a quarter of the thermal energy associated with the cathode.

A beam originating from a cathode which is then accelerated therefore tends to have a very small longitudinal temperature. For beams in which collisions are important there is a tendency for the transverse energy to be scattered into the longitudinal direction. This effect can be troublesome in electron microscope beams, where there is a high intensity associated with beam crossover in the focusing systems; it is known as the Boersch effect (BOERSCH, 1954, LOEFFLER, 1970). It is also of importance in charged particle storage rings, where it is sometimes known as the Touschek effect (BRUCK, 1966).

11. CONCLUSION

Paraxial beams with rather special velocity distributions have been studied both from hydrodynamical and optical points of view. In most practical situations only one of the descriptions will be appropriate; indeed, for many high current beams where time varying longitudinal fields are important the optical approach is clearly inadequate. Nevertheless it is instructive to see explicitly how the two approaches merge, and in particular explore the not entirely obvious relation between pressure and emittance.

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APPENDIX

Fields seen in a rotating frame in a uniform magnetic field

In a uniform magnetic field B_z the forces on a moving particle may be written

$$F_r = m(\ddot{r} - r\dot{\theta}_1^2) = qB_{z_1}r\dot{\theta}_1, \tag{A1}$$

$$F_\theta = m(r\ddot{\theta}_1 + 2\dot{r}\dot{\theta}_1) = -qB_{z_1}\dot{r}. \tag{A2}$$

In a frame of reference rotating with ω_f, the angle θ_2 is related to the angle θ_1 in the stationary frame by the expression $\theta_1 = \theta_2 + \omega_f t$. Substituting in equations (A1) and (A2), and writing Ω_L for $-qB_{z_1}/m$, yields for the forces in the new frame of reference

$$F_{r_2} = m(\ddot{r} - r\dot{\theta}_2^2) = -mr\omega_f(2\Omega_L - \omega_f) + 2mr\dot{\theta}_2(\Omega_L - \omega_f) \tag{A3}$$

$$F_{\theta_2} = m(r\ddot{\theta}_2 + 2\dot{r}\dot{\theta}_2) = 2m\dot{r}(\Omega_L - \omega_f). \tag{A4}$$

The force F_{r_2} contains one component which is independent of the particle velocity in the frame 2, and one component perpendicular to the velocity. These can be identified with E_{r_2} and B_{z_2}, to give

$$B_{z_2} = -\frac{2m}{q}(\Omega_L - \omega_f) = (1 - \omega_f/\Omega_L)B_{z_1}, \tag{A5}$$

$$E_{r_2} = \frac{m}{q}r\omega_f(-2\Omega_L + \omega_f) = -r\omega_f(1 - B_{z_1}\omega_f/2\Omega_L). \tag{A6}$$

For the special case when $\omega_f = \Omega_L$, the equations reduce to

$$B_{z_2} = 0 \qquad E_{r_2} = -\tfrac{1}{2}r\Omega_L B_{z_1}. \tag{A7}$$

If two species of particle with different mass are present, then it is only possible to arrive at the simple condition (A7) for one of them. If the second species represents background neutralization, this limitation is not troublesome.

This transformation is limited to velocities which are non-relativistic in the transverse direction. The velocity in the z direction may be relativistic, if so, m is set equal to γm_0.