# Review

The theory and practice of charged particle beam emittances have been a topic of interest since the early days of charged particle accelerators.

The concept of emittance has been widely used to define a figure of merit for the quality of charged particle beams.

## *Brightness*, a function of the emittance

The quantity brightness was adopted from conventional optics where it characterizes the quality of light sources. In charged particle beam applications, beam brightness is the current density per unit solid angle in the axial direction. Bright beams have high current density and good parallelism.

$$B \cong I / \pi^2 \epsilon^2$$

beam has current $I$,

# Emittance

**Charged particle beams may be looked at from two points of view.**

- **Optical approach**

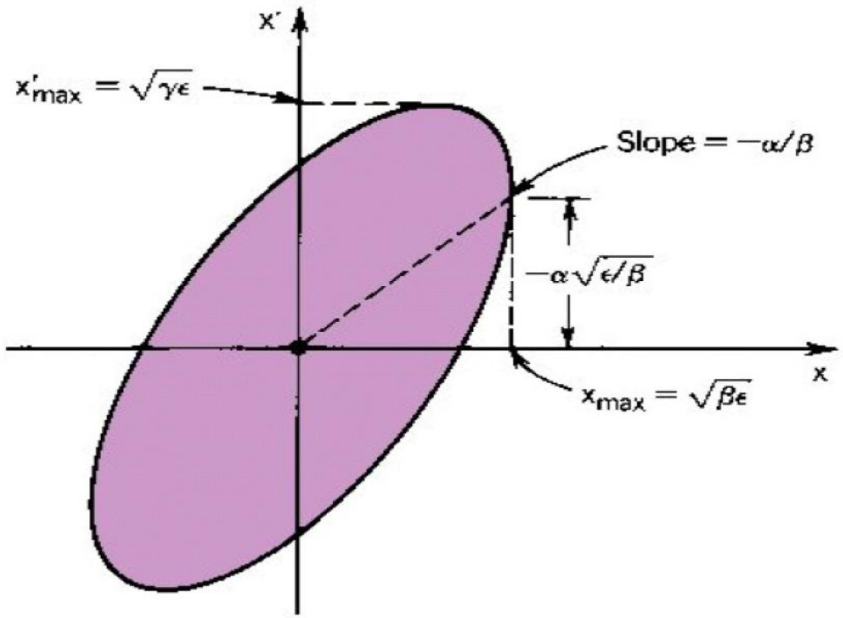
  **The beam is considered as a bundle of orbits, whose properties obey optical laws, derived from the principles of classical mechanics.**

- **Hydrodynamic description**

  **The beam may be considered as a hot gas, confined by external focusing fields, for which a hydrodynamic description is appropriate.**

# Review

## Optical approach




The ellipse equation is written as

$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_x$$

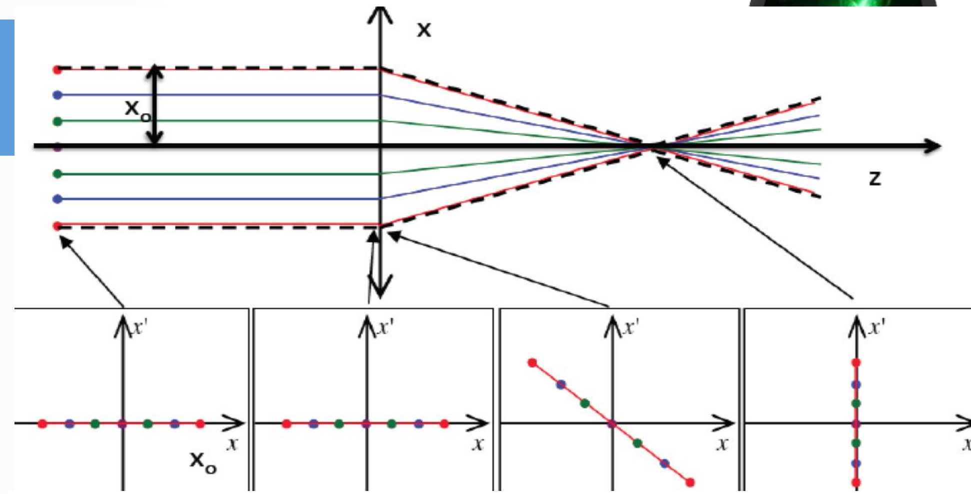
$$\beta_x \gamma_x - \alpha_x^2 = 1$$

Phase space distribution in a skewed elliptical boundary showing the relationship of Twiss parameters to the ellipse geometry .

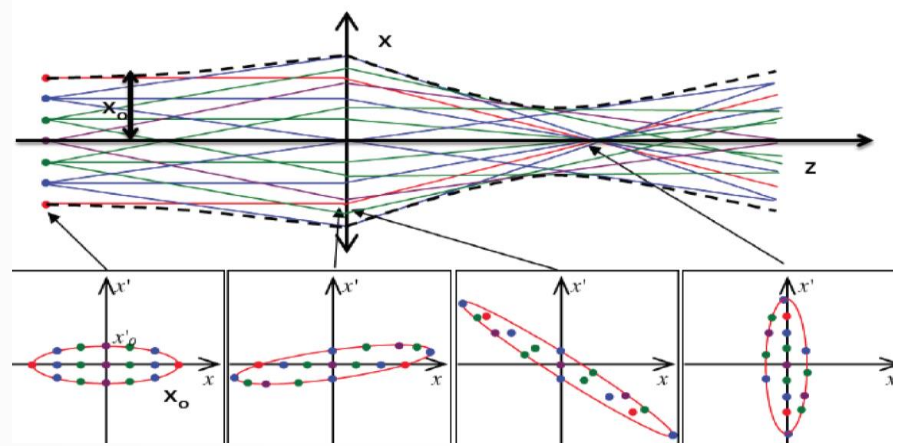
**According to Liouville's theorem the 6D (*x,px,y,py,z,pz*) phase space volume occupied by a beam is constant, provided that there are no dissipative forces, no particles lost or created, and no coulomb scattering among particles.**

# Charged Particle Beam

## Laminar




## Non Laminar




- ✓ Roughly cylindrical collection of charges, in which all the members of one species have a component of velocity of the same sign parallel to the axis.
- ✓ Sometimes only one type of particle is present, though often the system is partly or fully ‘neutralized’ by approximately stationary particles of the opposite sign.
- ✓ In many practical beams the velocity component parallel to the axis greatly exceeds the transverse velocity, and that the energy spread of the moving particles is small, a few percent or less

## Collisionless Boltzmann Equation: Vlasov Equation

$$\frac{Df\,(x,v_x,y,v_y,z,v_z)}{Dt} = \frac{\partial f}{\partial t} + v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y} + v_z\frac{\partial f}{\partial z} + a_x\frac{\partial f}{\partial v_x} + a_y\frac{\partial f}{\partial v_x} + a_z\frac{\partial f}{\partial v_x} = 0$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{a}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0.$$

The acceleration is given by the Lorentz expression:

$$\boldsymbol{a} = q\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,/\,m_o\,.$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{q}{m_o}\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0.$$

***Vlasov equation, a fundamental relationship in plasma physics.***

**Derive a set of reduced equations from the Vlasov equation by taking weighted averages over velocity space**

The $k = 0$ moment over the Boltzmann equation has the form:

$$\int_{-\infty}^{+\infty} dv_x \frac{\partial f}{\partial t} + \int_{-\infty}^{+\infty} dv_x\, v_x \frac{\partial f}{\partial x} + \int_{-\infty}^{+\infty} dv_x\, a_x \frac{\partial f}{\partial v_x} = 0.$$

In summary, the $k = 0$ moment equation is:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\, \overline{v_x}) = 0$$

**The change in particle density with time at a particular location**

We shall derive another equation from the collisionless Boltzmann equation by taking moments with $k = 1$ (Eq. 2.96). The one-dimensional form is

$$\int_{-\infty}^{+\infty} dv_x\, v_x \frac{\partial f}{\partial t} + \int_{-\infty}^{+\infty} dv_x\, v_x^2 \frac{\partial f}{\partial x} + \int_{-\infty}^{+\infty} dv_x\, a_x\, v_x \frac{\partial f}{\partial v_x} = 0. \qquad (2.109)$$

$$\frac{\partial}{\partial t}(m_o\, n\, \overline{v_x}) = -\frac{\partial}{\partial x}(m_o\, n\, \overline{v_x}^2) - \frac{\partial}{\partial x}(m_o\, n\, \overline{\delta v_x^2})$$

## Conservation of momentum at spatial positions within the particle distribution

**The left hand side, is the time rate of change of momentum per unit volume, , at a location.**

**The terms on the right hand side contribute to the momentum change**

**the average momentum at a point can change by two processes:**
**1. An applied force may accelerate particles in the volume element.**
**2. Particles may leave the element and be replaced by new particles with a different average momentum(collective physics - it can occur only if a system contains many particles ).**

# Kinetic theory of gases

$$\frac{\partial}{\partial t}(m_o\, n\, \overline{v_x}) = -\frac{\partial}{\partial x}(m_o\, n\, \overline{v_x}^{\,2}) - \frac{\partial}{\partial x}(m_o\, n\, \overline{\delta v_x^2})$$

$$v_x = \overline{v_x} + \delta v_x$$

$\overline{v_x}$ average velocity

$\delta v_x$ deviation from the average

$$\overline{v_x^2} = \overline{v_x}^{\,2} + \overline{\delta v_x^2}$$

$$v_x^2 = 0$$

$$\frac{1}{2} m\delta v_x^2 = \frac{1}{2} KT$$

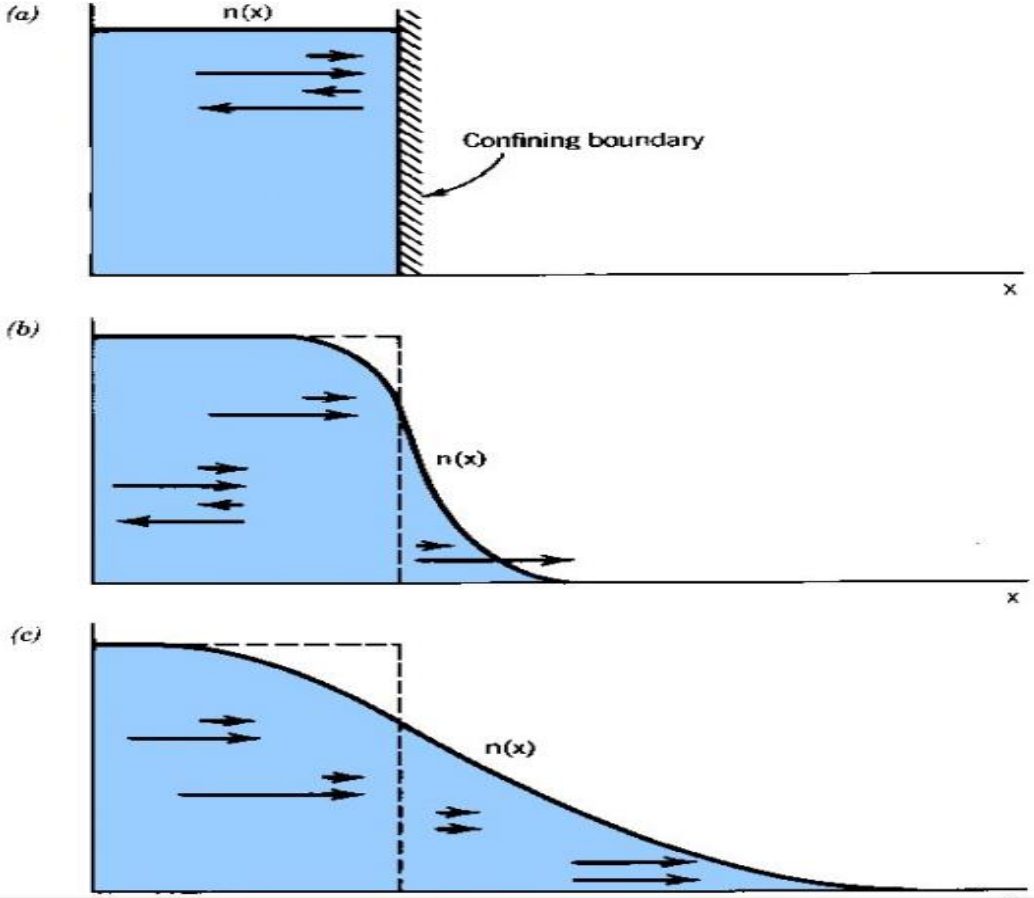
$$\frac{\partial}{\partial t}(m_0 n v_x) = -\frac{\partial}{\partial x}(nKT)$$

$$p = nKT$$

$$(\frac{\partial}{\partial t} m_0 n v_x) = -\frac{\partial}{\partial x}(p)$$

**Pressure Gradient force** $-\frac{\partial}{\partial x}(p) = -\nabla_x p$

**Both temperature and pressure depend on the velocity spread.**




**Entropy is a quantity which may be linked naturally to concepts such as the transverse temperature and pressure of the beam**

$$S/kN = S_0$$

$$\begin{aligned} S_0 &= \log N - \log AN/\pi\varepsilon \\ &= \log \pi\varepsilon - \log A. \end{aligned}$$

# Emittance Force

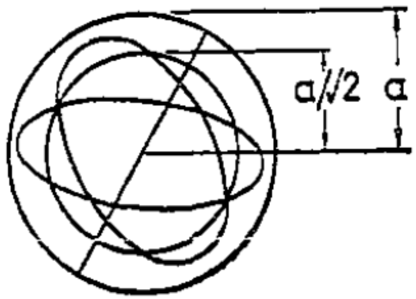



FIG. 3.—Projections on a plane perpendicular to the axis of some typical orbits in a uniform diameter beam with the microcanonical distribution defined by equation (30). The velocity of particles passing through any point on the projection is constant and isotropic, decreasing parabolically from a maximum value for the central point to zero for a point at the edge of the beam.

$$\frac{x^2 + y^2}{a^2} + \frac{x'^2 + y'^2}{\alpha^2} - 1 = 0.$$

**KAPCHINSKIJ and VLADIMIRS distribution**

**K-V is a continuous beam whose distribution projection is uniform in 2D sub phase spaces**

$$x'^2 + y'^2 = v_\perp{}^2/\beta^2 c^2, \qquad v^2 = \beta^2 c^2 \alpha^2 (a^2 - r^2)/a^2.$$

**This distribution function represents a three dimensional hyper-ellipsoidal shell in four dimensional xx' yy' space.**

**The density in the xy plane is uniform and bounded by a circle of radius a. In the xx' or yy' plane the projection is an ellipse with semi- axes a and alfha.**

# Emittance Force

$$\frac{x^2+y^2}{a^2}+\frac{x'^2+y'^2}{\alpha^2}-1=0.$$

$$x'^2+y'^2=v_\perp{}^2/\beta^2c^2,$$

$$v^2=\beta^2c^2\alpha^2(a^2-r^2)/a^2.$$

At any radius $(x^2+y^2)^{1/2}$, the value of $x'^2+y'^2$ is constant

The distribution at a point is isotropic $\pi_{rr}=\pi_{\theta\theta}$;

The transverse pressure is scalar $\Pi=\frac{1}{2}n_0\gamma m_0\beta^2c^2\alpha^2(a^2-r^2)/a^2.$

The pressure is a maximum on the axis and decreases to zero at the beam edge. The force per unit volume may be found:

$$\frac{\partial\Pi}{\partial r}=n_0\gamma m_0\beta^2c^2\alpha^2r/a^2.$$

## Emittance Force

**The force per unit volume may be found**

$$\frac{\partial \Pi}{\partial r} = n_0 \gamma m_0 \beta^2 c^2 \alpha^2 r / a^2.$$

**Normalized to a single particle this becomes, at r = a ,**

$$\varepsilon = a\alpha.$$

$$F_r = \gamma m_0 \beta^2 c^2 \alpha^2 / a = \gamma m_0 \beta^2 c^2 \varepsilon^2 / a^3$$

$$\frac{x^2 + y^2}{a^2} + \frac{x'^2 + y'^2}{\alpha^2} - 1 = 0.$$

The quantity $\varepsilon = a\alpha.$ is defined as the 'emittance' of the beam. It is seen to be $1/\pi$ times the area occupied by the trajectories projected on to the *xx'* or *yy'* plane.

# Envelope Equation

**We are now interested in following the evolution of particle distribution during beam transport and acceleration.**

$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_{x,\mathrm{rms}}$$

$$\sigma_{xx'}(z) = \langle x x' \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x x' f(x, x', z)\,\mathrm{d}x\mathrm{d}x' .$$

$$\sigma_x^2(z) = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x, x', z)\,\mathrm{d}x\mathrm{d}x'$$

$$\sigma_{x'}^2(z) = \langle x'^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x'^2 f(x, x', z)\,\mathrm{d}x\mathrm{d}x'$$

$$\sigma_x = \sqrt{\beta_x \varepsilon_{x,\mathrm{rms}}}$$

$$\sigma_{x'} = \sqrt{\gamma_x \varepsilon_{x,\mathrm{rms}}}$$

$$\frac{\mathrm{d}\sigma_x}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2 \rangle = \frac{1}{2\sigma_x} 2\langle x x' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{\mathrm{d}^2\sigma_x}{\mathrm{d}z^2} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{\mathrm{d}\sigma_{xx'}}{\mathrm{d}z} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle + \langle x x' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle x x'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle x x'' \rangle}{\sigma_x}$$

$$\sigma_x'' - \frac{1}{\sigma_x}\langle x x'' \rangle = \frac{\varepsilon_{\mathrm{rms}}^2}{\sigma_x^3} .$$

**The emittance term can be interpreted physically as an outward pressure on the beam envelope produced by the r.m.s. spread in trajectory angle, which is parameterized by the r.m.s. emittance.**

# Envelope Equation

**For the effective transport of a beam with finite emittance it is mandatory to make use of some external force providing beam confinement in the transport or accelerating line. The term *xx″* accounts for external forces when we know *x″* given by the single particle equation of motion:**

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = F_x.$$

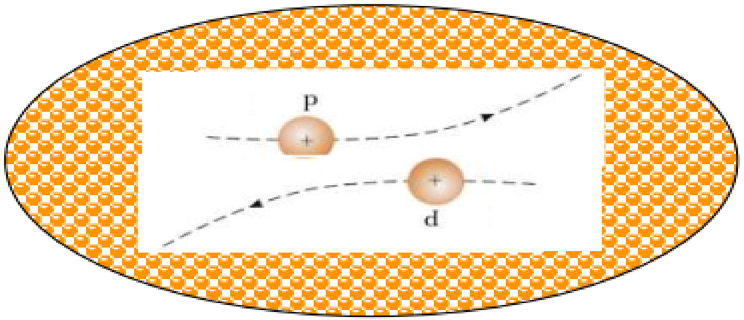
$$\sigma_x'' + \frac{p'}{p}\sigma_x' - \frac{1}{\sigma_x}\frac{\langle xF_x \rangle}{\beta c p} = \frac{\varepsilon_{n,\mathrm{rms}}^2}{\gamma^2 \sigma_x^3}$$

**Notice that the effect of longitudinal accelerations appears in the r.m.s. envelope equation as an oscillation damping term, called ‘adiabatic damping’, proportional to *p′/p*. The term *xFx* represents the moment of any external transverse force acting on the beam, such as that produced by a focusing magnetic channel**

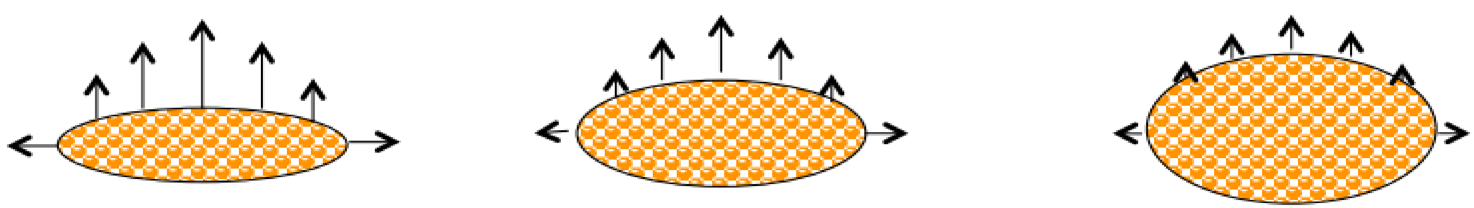
# Space charge forces

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**




2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



# Envelope Equation with Longitudinal Acceleration

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_{rms}}$$

# Bunch radius matched to the laser wakefield

$$\frac{d^2\sigma}{d\tau^2} - \frac{E_z}{\gamma}\frac{d\sigma}{d\tau} + \frac{f}{\gamma}\sigma - \frac{(k_p\varepsilon_n)^2}{\gamma^2\sigma^3} = 0.$$

$$\sigma_{\mathrm{match}} = (k_p\varepsilon_n)^{1/2}/(f\gamma)^{1/4},$$

$$\sigma_{\mathrm{match}} \approx 2\ \mu\mathrm{m},$$