Relativistic second-order dissipative hydrodynamics from Zubarev's non-equilibrium statistical operator

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Outline

- Introduction & motivation
- Non-equilibrium statistical operator (NESO) formalism
- Relativistic hydrodynamics equations
  - First-order hydrodynamics equations
  - Second-order hydrodynamics equations
- Summary & outlook

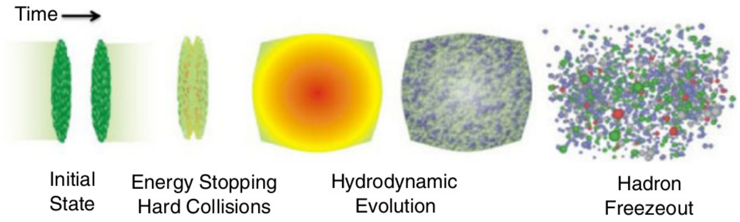
Applications of relativistic hydrodynamics

- Hydrodynamics is a powerful tool to describe low-frequency and long-wavelength phenomena in statistical systems.
- It finds numerous applications in nuclear physics, astrophysics and cosmology.
- Relativistic hydrodynamics has been successfully applied to describe the behavior of QGP created in heavy-ion collision experiments at RHIC and LHC.
- Binary neutron-star mergers and gravitational waves emitted in these events are modelled by relativistic hydrodynamics in general relativity.

Aim and novelty of the work

- In this work, we provide a new, alternative derivation of relativistic second-order dissipative hydrodynamics within Zubarev's formalism.
- This method is based on a generalization of the Gibbs canonical ensemble to non-equilibrium states.
- Advantage of this formalism is that the transport coefficients are automatically obtained in the form of Kubo-type relations, valid also in the strong-coupling limit.

Relativistic heavy-ion collisions



Dynamical stages of ultrarelativistic heavy-ion collisions

1. Pre-equilibrium
   At proper times $\tau \lesssim 1$ fm after the collision the distribution of particle momenta in the expanding fireball of QGP is not thermal.
2. Thermalization and hydrodynamic flow
   At time scales $1 \lesssim \tau \lesssim 4$ fm local thermal equilibrium is achieved, and the evolution of QGP can be described by relativistic fluid dynamics.
3. Hadronization and freeze-out
   At $\tau \gtrsim 5$ fm QGP undergoes a phase transition to the hadronic phase and goes out of equilibrium. In this stage the particle flows should be described by kinetic theory.

Relativistic hydrodynamics (fluid dynamics)

Fluid dynamics is an effective theory which describes collective phenomena in many-particle systems in low-frequency and long-wave-length limit.

- Relativistic fluids are described completely by the energy-momentum tensor $T^{\mu\nu}$ and conserved particle 4-current N^{μ} – 14 variables

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^{\mu} = 0 \qquad - \qquad \text{5 equations.} \qquad (1)$$

- $T^{\mu\nu}$ and N^{μ} can be decomposed as ($\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$)

$$T^{\mu\nu} = \underbrace{\epsilon u^\mu u^\nu - p\Delta^{\mu\nu}}_{\textit{ideal part}} - \underbrace{\Pi\Delta^{\mu\nu} + \pi^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu}_{\textit{dissipative part}}, \qquad N^\mu = \underbrace{n u^\mu}_{\textit{id.}} + \underbrace{j^\mu}_{\textit{diss.}}$$

-

equilibrium quantities				dissipative quantities			
u^μ	–	fluid 4-velocity	[3]	$\pi^{\mu\nu}$	–	shear stress tensor	[5]
ϵ	–	energy density	[1]	q^μ	–	energy diffusion flux	[3]
n	–	particle density	[1]	j^μ	–	particle diffusion flux	[3]
p	–	eq. pressure, fixed by EOS		Π	–	bulk viscous pressure	[1]

- The system (1) should be closed by an **equation of state $p = p(\epsilon, n)$** + **9 additional equations** for the dissipative quantities.

Equations of relativistic hydrodynamics

- The fluid 4-velocity u_μ should be connected to one of the physical currents:
  1. Landau-Lifshitz frame: u_μ is connected to the energy flow
  $$\epsilon = \sqrt{u_\nu T^{\mu\nu} u^\lambda T_{\mu\lambda}}, \qquad u_\nu T^{\mu\nu} = \epsilon u^\mu, \qquad q^\mu = 0.$$
  2. Eckart frame: u_μ is connected to the particle flow
  $$n = \sqrt{N^\mu N_\mu}, \qquad N^\mu = n u^\mu, \qquad j^\mu = 0.$$
- The conservation laws for the energy-momentum tensor and particle current lead to the equations of relativistic hydrodynamics (Landau frame)
$$\begin{aligned} Dn + n\theta + \partial_\mu j^\mu &= 0, \\ D\epsilon + (h+\Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} &= 0, \\ (h+\Pi)Du_\alpha - \nabla_\alpha(p+\Pi) + \Delta_{\alpha\nu}\partial_\mu \pi^{\mu\nu} &= 0. \end{aligned}$$
- Equations of ideal hydrodynamics read ($h = \epsilon + p$ is the enthalpy density)
$$Dn + n\theta = 0, \qquad D\epsilon + h\theta = 0, \qquad hDu_\alpha = \nabla_\alpha p.$$

Dissipative fluid dynamics

1. **Ideal fluid dynamics**
   Based on the assumption of *local thermal equilibrium*
   $\Rightarrow$ no dissipation: $\Pi = \pi^{\mu\nu} = q^{\mu} = j^{\mu} = 0$.
2. **Relativistic Navier-Stokes (first-order) theory**
   Linear constitutive relations beween dissipative fluxes and thermodynamic forces
   $\Rightarrow$ suffers from acausality and numerical instability [1]
3. **Müller–Israel-Stewart (second-order) theory**
   Relaxation-type equations for dissipative quantities $\Rightarrow$ causality is restored.

Unlike the first-order hydrodynamics, where the equations of motion have a unique form, the equations of the second-order hydrodynamics are not well established so far.

Various methods to derive transport equations and transport coefficients

- Kinetic theory: based on the Boltzmann equation for the quasiparticle distribution function $\Rightarrow$ applicable for *systems with well-defined quasiparticles*.
- **Kubo-Zubarev formalism**: based on the quantum Liouville equation for the statistical operator (density matrix) $\Rightarrow$ applicable for *strongly interacting systems*.
- Holographic methods: based on fluid/gravity duality $\Rightarrow$ applicable for certain class of field theories in the *limit of infinitely strong coupling*.

[1] Recently it was shown that these acausalities and instabilities are a consequence of the matching procedure to the local-equilibrium reference state.

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Local equilibrium statistical operator

- Thermodynamics of quantum systems is described via the statistical operator $\hat{\rho}(t)$ (density matrix), which in thermal equilibrium is given by the Gibbs distribution

$$\hat{\rho}_{\rm eq} = Z^{-1} \exp\Big[-\beta\Big(\hat{H} - \mu\hat{\mathcal{N}}\Big)\Big].$$

- In an arbitrary reference frame we replace $\hat{H} \to \hat{\mathcal{P}}_\nu u^\nu$, and substitute

$$\hat{\mathcal{P}}_\nu = \int d^3x\, \hat{T}_{0\nu}(x), \qquad \hat{\mathcal{N}} = \int d^3x\, \hat{N}_0(x).$$

- The Lorentz-covariant form of the Gibbs distribution reads ($x \equiv (\boldsymbol{x}, t)$)

$$\hat{\rho}_{\rm eq} = Z^{-1} \exp\Big\{-\int d^3x\, \beta\Big[u^\nu \hat{T}_{0\nu}(x) - \mu\hat{N}_0(x)\Big]\Big\}.$$

- In *local thermal equilibrium* $\beta \to \beta(x)$, $\mu \to \mu(x)$, $u_\mu \to u_\mu(x)$, and

$$\hat{\rho}_{\rm eq}(\beta, \mu, u_\mu) \to \hat{\rho}_l\Big[\beta(x), \mu(x), u_\mu(x)\Big].$$

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Local equilibrium distribution cannot describe irreversible processes!

Solving the Liouville equation

The NESO $\hat{\rho}(t)$ should be found from the Liouville equation (Heisenberg picture)

$$\frac{d\hat{\rho}(t)}{dt} = 0.$$

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2. The required *irreversible* solution for NESO can be obtained by averaging the solution above over all possible initial states from $-\infty < t_0 < t$

$$\hat{\rho}(t) \to \lim_{T\to\infty} \frac{1}{T+t} \int_{-T}^{t} dt_0 \, \hat{\rho}_l(t_0) = \lim_{\varepsilon\to+0} \varepsilon \int_{-\infty}^{t} dt_0 \, e^{\varepsilon(t_0-t)} \hat{\rho}_l(t_0),$$

where we used Abel's theorem.

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This operator satisfies the Liouville equation with an infinitesimal source term

$$\frac{d\hat{\rho}(t)}{dt} = -\lim_{\varepsilon\to+0} \varepsilon \Big[\hat{\rho}(t) - \hat{\rho}_l(t)\Big].$$

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Non-equilibrium statistical operator

Now the full NESO can be written as

$$\hat{\rho}(t) = Q^{-1} e^{-\hat{A}+\hat{B}}, \qquad Q = \mathrm{Tr} e^{-\hat{A}+\hat{B}},$$

where

$$\hat{A}(t) = \int d^3x \left[\beta^{\nu}(x)\hat{T}_{0\nu}(x) - \alpha(x)\hat{N}_0(x) \right]$$

is the local equilibrium part with $\beta^{\nu} = \beta u^{\nu}$, $\alpha = \beta\mu$, and

$$\begin{aligned} \hat{B}(t) &= \lim_{\varepsilon \to +0} \int d^3x_1 \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1 - t)}\, \hat{C}(x_1), \\ \hat{C}(x) &= \hat{T}_{\mu\nu}(x)\partial^{\mu}\beta^{\nu}(x) - \hat{N}_{\mu}(x)\partial^{\mu}\alpha(x). \end{aligned}$$

The operator $\hat{B}$ is a thermodynamic "force" as it involves the gradients of β, μ and u_{μ}. Naturally, it can be identified with the non-equilibrium part of the statistical operator.

If $\hat{B} = 0$, we recover the local equilibrium statistical operator

$$\hat{\rho}_l(t) = Q_l^{-1} e^{-\hat{A}}, \qquad Q_l = \mathrm{Tr} e^{-\hat{A}}.$$

Statistical averages and correlation functions

- Now we treat the non-equilibrium part $\hat{B}$ of the NESO as a perturbation

$$\hat{\rho} = \hat{\rho}_l + \hat{\rho}_1 + \hat{\rho}_2.$$

- Statistical average of any operator $\hat{X}(x)$ can be written as

$$\langle \hat{X}(x) \rangle = \mathrm{Tr}\big[\hat{\rho}(t)\hat{X}(x)\big] = \langle \hat{X}(x) \rangle_l + \int d^4x_1 \Big(\hat{X}(x), \hat{C}(x_1)\Big) + \int d^4x_1 d^4x_2 \Big(\hat{X}(x), \hat{C}(x_1), \hat{C}(x_2)\Big).$$

- This is actually a power series in the Knudsen number $\mathrm{Kn} = \lambda/L \ll 1$, which is the ratio of typical *microscopic* length scale λ over which the correlation functions decay to the typical *macroscopic* length scale L over which the hydrodynamic parameters vary in space (*i.e.*, the thermodynamic force $\hat{C}(x) \propto 1/L$).

- The two-point correlation function is related to the retarded Green's function

$$\Big(\hat{X}(x), \hat{C}(x_1)\Big) = -\frac{1}{\beta} \int_{-\infty}^{t_1} dt' \, G^R_{\hat{X}\hat{C}}(\boldsymbol{x} - \boldsymbol{x}_1, t - t').$$

First-order transport equations

- The thermodynamic force to the first-order in gradients can be decomposed as

$$\hat{C} = -\beta\hat{p}^*\theta + \beta\hat{\pi}_{\mu\nu}\sigma^{\mu\nu} + h'^{-1}\hat{h}_\mu\nabla^\mu\alpha,$$

where $\theta = \partial_\mu u^\mu$, $\nabla_\mu = \Delta_{\mu\nu}\partial^\nu$, $\sigma_{\mu\nu} = \nabla_{\langle\alpha}u_{\beta\rangle}$, $h' = (\epsilon + p)/n$, $\hat{h}_\mu = \hat{q}_\mu - h\hat{j}_\mu$, $\hat{p}^* = \hat{p} - \frac{\partial p}{\partial\epsilon}\hat{\epsilon} - \frac{\partial p}{\partial n}\hat{n}$.

- In this approximation we obtain the relativistic NS equations (Landau frame)

$$\Pi = -\zeta\theta, \qquad \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu}, \qquad h_\mu = -\kappa T^2 h'^{-1}\nabla_\mu\alpha.$$

These equations are of first order in the Knudsen number $\mathrm{Kn} = \lambda/L \ll 1$.

- Three (first-order) transport coefficients:

ζ – bulk viscosity

η – shear viscosity

κ – thermal conductivity

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- Three (first-order) transport coefficients:

ζ – bulk viscosity

η – shear viscosity

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- Navier-Stokes equations imply **instantaneous** response of dissipative fluxes to thermodynamic forces $\Rightarrow$ **causality is violated!** $\Rightarrow$ **second-order theory is required**

Kubo formulas

The first-order transport coefficients are given by the following Green-Kubo relations

$$\eta = -\frac{1}{10}\frac{d}{d\omega}\mathrm{Im}G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\bigg|_{\omega=0},$$

$$\zeta = -\frac{d}{d\omega}\mathrm{Im}G^R_{\hat{p}^*\hat{p}^*}(\omega)\bigg|_{\omega=0},$$

$$\kappa = \frac{\beta}{3}\frac{d}{d\omega}\mathrm{Im}G^R_{\hat{h}_\mu\hat{h}^\mu}(\omega)\bigg|_{\omega=0},$$

where

$$G^R_{\hat{X}\hat{Y}}(\omega) = -i\int_0^\infty dt e^{i\omega t}\int d^3x\langle\big[\hat{X}(x),\hat{Y}(0)\big]\rangle_l$$

is the two-point equilibrium retarded Green's function.

Second-order transport equations

The final transport equations in the Landau frame read

$$
\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi &= -\zeta\theta + \tilde{\lambda}_\Pi \theta\Pi + \zeta_1 \sigma_{\mu\nu}\pi^{\mu\nu} + \zeta_2 \partial_\mu j^\mu \\
&\quad + \varsigma\theta^2 - \lambda_{\Pi\pi}\sigma_{\mu\nu}\sigma^{\mu\nu} + T\zeta_\Pi \nabla_\mu \alpha \nabla^\mu \alpha, \\
\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} &= 2\eta\sigma_{\mu\nu} + \tilde{\lambda}_\pi \theta \pi_{\mu\nu} + \lambda_\pi \sigma_{\rho<\mu}\sigma^\rho_{\nu>} \\
&\quad + \lambda_{\pi\Pi}\theta\sigma_{\mu\nu} + \lambda_{\pi j}\nabla_{<\mu}\alpha\nabla_{\nu>}\alpha, \\
\tau_j \dot{j}_\mu + j_\mu &= \chi\nabla_\mu\alpha + \tilde{\lambda}_j \theta j_\mu + \chi^*\theta\nabla_\mu\alpha - \lambda_{j\pi}\sigma_{\mu\nu}\nabla^\nu\alpha \\
&\quad + \tilde{\chi}\left(\Pi\dot{u}_\mu - \nabla_\mu\Pi + \Delta_{\mu\sigma}\partial_\nu\pi^{\nu\sigma}\right),
\end{aligned}
$$

where $\dot{\Pi} = D\Pi, \quad \dot{\pi}_{\mu\nu} = D\pi_{<\mu\nu>}, \quad \dot{j}_\mu = \Delta_{\mu\nu} D j^\nu, \quad \dot{u}_\mu = D u_\mu, \quad D = u^\nu\partial_\nu.$

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$$\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \tilde{\lambda}_\pi \theta \pi_{\mu\nu} + \lambda_\pi \sigma_{\rho<\mu}\sigma^\rho_{\nu>} + \lambda_{\pi\Pi}\theta\sigma_{\mu\nu} + \lambda_{\pi j}\nabla_{<\mu}\alpha\nabla_{\nu>}\alpha,$$

$$\tau_j \dot{j}_\mu + j_\mu = \chi\nabla_\mu \alpha + \tilde{\lambda}_j \theta j_\mu + \chi^* \theta \nabla_\mu \alpha - \lambda_{j\pi}\sigma_{\mu\nu}\nabla^\nu \alpha + \tilde{\chi}\left(\Pi \dot{u}_\mu - \nabla_\mu \Pi + \Delta_{\mu\sigma}\partial_\nu \pi^{\nu\sigma}\right),$$

where $\dot{\Pi} = D\Pi, \quad \dot{\pi}_{\mu\nu} = D\pi_{<\mu\nu>}, \quad \dot{j}_\mu = \Delta_{\mu\nu} D j^\nu, \quad \dot{u}_\mu = D u_\mu, \quad D = u^\nu \partial_\nu.$

Transport equations imply relaxation of dissipative fluxes to their NS values

$$\tau_\Pi \dot{\Pi} + \Pi = 0 \quad \Rightarrow \quad \Pi(t) \propto \exp(-t/\tau_\Pi) \quad \text{"transient modes"}$$

The response of the system to thermodynamic perturbations is now on finite time scales τ_Π, τ_π and τ_j $\Rightarrow$ **causality can be restored!**

Second-order transport coefficients

The relaxation times τ_π, τ_Π and τ_h are given by Kubo-type formulas

$$\tau_\Pi = \frac{1}{2\zeta}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{p}^*\hat{p}^*}(\omega)\Big|_{\omega=0},$$

$$\tau_\pi = \frac{1}{20\eta}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\Big|_{\omega=0},$$

$$\tau_j = -\frac{\beta}{6\kappa}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{h}_\mu\hat{h}^\mu}(\omega)\Big|_{\omega=0}.$$

Nonlinear transport coefficients involve three-point correlation functions

$$\lambda_\pi = \frac{12}{35}\beta^2\int d^4x_1 d^4x_2\Big(\hat{\pi}^\delta_\gamma(x), \hat{\pi}^\lambda_\delta(x_1), \hat{\pi}^\gamma_\lambda(x_2)\Big),$$

$$\lambda_{\pi\Pi} = -\frac{\beta^2}{5}\int d^4x_1 d^4x_2\Big(\hat{\pi}_{\gamma\delta}(x), \hat{\pi}^{\gamma\delta}(x_1), \hat{p}^*(x_2)\Big),$$

$$\lambda_{\pi j} = \frac{1}{5}\int d^4x_1 d^4x_2\Big(\hat{\pi}_{\gamma\delta}(x), \hat{J}^\gamma(x_1), \hat{J}^\delta(x_2)\Big).$$

These coefficients account for nonlinear couplings between different dissipative processes.

Summary

- We provided a novel derivation of second-order relativistic fluid dynamics using Zubarev's method of a non-equilibrium statistical operator.
- We derived transport equations for the shear stress tensor, the bulk viscous pressure and the diffusion/heat current.
- We obtained formal expressions for the second-order transport coefficients in terms of certain two- and three-point equilibrium correlation functions.

Future outlook

- Computation of second-order transport coefficients for strongly coupled QCD.
- Inclusion of gauge fields and/or quantum anomalies.
- Inclusion of vorticity terms important for phenomenology of heavy-ion collisions.

THANK YOU FOR ATTENTION!