

RELATION BETWEEN FIELD ENERGY AND RMS EMITTANCE IN INTENSE PARTICLE BEAMS*

T. P. Wangler, K. R. Crandall, R. S. Mills, and M. Reiser,[†] AT-1, MS H817
 Los Alamos National Laboratory, Los Alamos, NM 87545 USA
[†]University of Maryland, College Park, MD 20742, USA

Summary

An equation is presented for continuous beams with azimuthal symmetry and continuous linear focusing, which expresses a relationship between the rate of change for squared rms emittance and the rate of change for a quantity we call the nonlinear field energy. The nonlinear field energy depends on the shape of the charge distribution and corresponds to the residual field energy possessed by beams with nonuniform charge distributions. The equation can be integrated for the case of an rms matched beam to yield a formula for space-charge-induced emittance growth that we have tested numerically for a variety of initial distributions. The results provide a framework for discussing the scaling of rms emittance growth and an explanation for the well-established lower limit on output emittance.

Introduction

A quantitative understanding of emittance growth induced by space charge is of great importance for the design of high-current linear accelerators and beam transport systems. The first systematic numerical study of emittance growth for linac beams was done by Chasman,¹ and more recent studies were reported by Jameson and Mills.² An interest in beam instabilities of periodic transport channels was stimulated by heavy-ion fusion requirements, and led to work recently reported by Hofmann, Laslett, Smith, and Haber.³ The mechanism of equipartitioning and emittance growth in linacs was studied by Hofmann and Bozsik⁴ and also by Jameson.⁵ Reviews of this work have been presented by Jameson⁶ and Hofmann.⁷

In spite of these efforts, basic phenomena observed both in numerical studies and in unpublished experimental data have remained unexplained, and no clear explanation of many of the important effects has emerged. Until recently, little attention has been given to a comparison of rms emittance growth for different initial distributions. Recently, numerical studies were published by Struckmeier, Klabunde, and Reiser⁸ for a continuous beam in a periodic channel, which led them to observe that some distributions experience a rapid initial emittance growth that increases with beam intensity. These authors hypothesized that the observed emittance growth was associated with a homogenization of the charge density and resulted in a conversion of space-charge field energy into transverse particle energy. They suggested that transverse energy conservation could be used to obtain a useful formula for emittance growth. Two formulas were presented, which were interpreted as upper and lower bounds on the emittance growth.

The idea of space-charge field energy as a useful quantity has been suggested before. In 1970, Gluckstern, et al.⁹ used electric-field energy comparisons to investigate the relative stability of different stationary distributions. Electric-field energy relationships were considered in more detail by Lapostolle^{10,11} in his important work in 1971, which, together with work by Sacherer,¹² resulted in a generalized rms envelope equation. Thus, stimulated by the suggestions of Struckmeier, Klabunde, and Reiser, we have attempted to extend the initial work of Lapostolle

and Sacherer, to investigate further the relationship between space-charge field energy and rms emittance for continuous beams in continuous focusing channels.

RMS Emittance and Nonlinear Field Energy

We consider the problem of a continuous, azimuthally symmetric beam that propagates in the +z-direction at constant velocity v . We assume that the beam is confined radially by a continuous, linear, external focusing force and that the paraxial approximation is valid in which all particles have the same longitudinal velocity $v \gg v_t$, where v_t is the transverse velocity component. We consider the characteristics of a steady-state solution, where the charge density, current density, and fields have no explicit dependence on time. We allow for initial phase-space distributions that are not necessarily stationary, and in general, we expect that the charge density $\rho(r, z)$ will evolve from an initial state at $z = 0$ to some final state, which may be stationary or independent of z .

Maxwell's equations can be written using nonzero self-field components E_r , E_z , and B_θ , and current density components j_r and j_z . A solution is easily obtained in the approximation that $\partial E_z / \partial z \ll \rho / \epsilon_0$, where ϵ_0 is the free-space permittivity, which leads to

$$B_\theta(r, z) = \frac{v}{c} E_r(r, z) \quad (1)$$

and

$$j_r(r, z) = -\epsilon_0 v \frac{\partial E_r(r, z)}{\partial z} \quad (2)$$

The component $E_z(r, z)$ is not ignored, but is determined by the requirement that $\oint \mathbf{E} \cdot d\mathbf{s} = 0$, and by a boundary condition such as a perfectly conducting pipe that encloses the beam.

The second moments of the charge distribution in x, x' phase space are $\overline{x^2}$, $\overline{x'^2}$ and $\overline{xx'}$. It can be shown that the derivatives of the moments are functions of the moments and of the values of $x F_x$ and $x' F_x$, where F_x is the x-component of the total force, the sum of the external plus the self-force.^{11,12} The definition of rms emittance ϵ in x, x' phase space, given by Lapostolle, can be shown to correspond to the total emittance of a continuous equivalent uniform beam (same second moments as the real beam) and is given by

$$\epsilon = 4(\overline{x^2} \overline{x'^2} - \overline{xx'}^2)^{1/2} \quad (3)$$

The rms emittance represents a convenient measure of the macroscopic or effective emittance of a beam subjected to nonlinear forces that arise either from external or self-fields.¹³ If Eq. (3) is differentiated and if the external force is assumed to be linear, it can be shown that

$$\frac{d\epsilon^2}{dz} = \frac{32}{m\gamma^2} (\overline{x^2} \overline{x' F_{sx}} - \overline{xx'} \overline{x F_{sx}}) \quad (4)$$

where m is the mass, and $\gamma = (1 - v^2/c^2)^{-1/2}$. For beam particles with charge e , $F_{sx} = eE_x/\gamma^2$ is the x-component of the self-force, including both electric and magnetic terms, which have opposite signs. Sacherer derived $x F_{sx}$ for an arbitrary charge distribution with elliptical symmetry, and his result can be written

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$$\overline{x'F_s} = 2w_0/N\gamma^2, \quad (5)$$

where N is number of particles per unit length and

$$w_0 = (eN)^2/16\pi\epsilon_0. \quad (6)$$

The value of $\overline{x'F_{sx}}$ can be derived using Eq. (2), which leads to

$$\overline{x'F_{sx}} = -\frac{1}{2N\gamma^2} \frac{dW}{dz}, \quad (7)$$

where

$$W = \pi\epsilon_0 \int_0^\infty r E_r^2 dr \quad (8)$$

is the self-electric field energy per unit length associated with the radial electric field. With an infinite value for the upper limit to the integral of Eq. (8), we find that the field energy W diverges. However, only field energy changes or field energy differences will be important. Therefore, because the electric field outside the beam is independent of the charge density, the infinite upper limit can be replaced by any finite radius larger than the actual beam. For a uniform beam, we write the field energy as W_U , and for a volume within a radius b (larger than the beam) we obtain

$$W_U = w_0(1 + 4 \ln b/X), \quad \text{and } b \geq X, \quad (9)$$

where X , defined by $X = 2\sqrt{x^2}$ is the total beam radius for a uniform beam.

Using these results, Eq. (4) can be written as

$$\frac{d\epsilon^2}{dz} = -\frac{X^2 K}{2} \frac{d}{dz} \left(\frac{U}{w_0} \right), \quad (10)$$

where X (twice the rms beam size) is interpreted as the total beam radius of the equivalent uniform beam, K is the generalized perveance given by

$$K = \frac{eI}{2\pi\epsilon_0 m v^3 \gamma^3}, \quad (11)$$

and

$$U = W - W_U. \quad (12)$$

We learned recently that a form equivalent to Eq. (10) was included by Lapostolle in Refs. 10 and 11. Instead of U/w_0 , Lapostolle's equation uses a field-energy correction term, ϵ^2 , which can be shown to be linearly related to U/w_0 . We also learned that a similar equation was obtained by Lee, Yu, and Barletta¹⁴ for a beam with no external focusing.

The quantity U is the difference between the self-electric field energies per unit length of the distribution and of the equivalent uniform beam. The special role of the uniform distribution can be explained by its associated linear self-force, which causes no rms emittance growth. Thus, U is the residual self-electric field energy possessed by beams with nonuniform charge distributions. Because nonuniform beams have nonlinear self-fields, we call U the nonlinear field energy. Equation (10) implies that a

decrease in nonlinear field energy U corresponds to an increase in rms emittance. Both the electric- and magnetic-field contributions are contained in Eq. (10) by including the factor γ^3 in the definition of K (γ^2 accounts for the magnetic field and γ accounts for relativistic mass). Although the total electromagnetic stored energy is the sum of electric plus magnetic terms, it is the difference that is related to changes in transverse rms emittance. This is because the transverse electric and magnetic self-force terms have opposite signs. Therefore, rms emittance growth, induced by space charge, is inherently a nonrelativistic effect; it is most important when γ is near unity.

From Eq. (9), w_0 is the self-electric field energy per unit length within the beam boundary of an equivalent uniform beam. The quantity U/w_0 is dimensionless, and its properties can easily be demonstrated by considering the example of a power-law charge distribution. We find that U/w_0 is zero for a uniform charge distribution and is positive both for peaked and hollow distributions, increasing as the distribution becomes more nonuniform. Furthermore, U/w_0 is independent of both beam current and rms beam size, and is a function only of the shape of the distribution. Thus, Eq. (10) shows that rms emittance changes are associated with three separate factors: rms beam size, perveance, and changes in shape of the charge distribution. Values of U/w_0 are given for example distributions in Table I. These distributions are listed in order from most peaked to most hollow.

TABLE I
U/w₀ FOR SOME COMMON DISTRIBUTIONS

Distribution Function	Charge Density $\rho(r)$	U/w_0
Gaussian	$\exp(-r^2/a)$	0.154
Waterbag	$1 - (r/R)^2$	$r \leq R$ 0.0224
Uniform	1	$r \leq R$ 0.000
Hollow ($n = 2$)	r^2	$r \leq R$ 0.0754
Hollow ($n = 10$)	r^{10}	$r \leq R$ 0.245

Equation (10) can be easily integrated, if we assume an rms-matched beam with constant X . The resulting relation between rms emittance and nonlinear field energy can be expressed as

$$\frac{\epsilon}{\epsilon_i} = \left[1 - \frac{(U - U_i)}{2w_0} \left(\frac{\omega_0^2}{\omega_i^2} - 1 \right) \right]^{1/2}, \quad (13)$$

where ω_0 is the zero-current betatron frequency, ω_i is the initial betatron frequency for the equivalent uniform beam (including space charge), and ϵ_i and U_i are initial values of rms emittance and nonlinear field energy. In Eq. (13), the rms emittance is expressed as the product of two factors, one, $(U - U_i)/w_0$, related to the change in the shape of the distribution, and one related to the initial betatron tune ratio ω_i/ω_0 . It is easy to show that the tune ratio is $\omega_i/\omega_0 = \sqrt{1 + u^2} - u$, where a space-charge parameter $u = Kv/2\epsilon_i\omega_0$.

We also find that an expression for transverse energy conservation can be derived using the assumptions given above. We obtain $\bar{T} + V_e + W/N\gamma^2 = \text{constant}$, where \bar{V}_e is the average potential energy associated with the external force, and the average transverse kinetic energy is $\bar{T} = m\gamma^2 \overline{x'^2}$ for an azimuthally symmetric beam in the paraxial approximation. In the limit of a perfect rms matched beam, \bar{V}_e and W_U are constant, so that a decrease in U (and therefore W) corresponds to an increase in \bar{T} . In general, an initial rms-matched beam will not remain exactly matched if the rms emittance grows.

Numerical Studies

We have made numerical studies using 1000 beam particles per calculation to investigate the effects described above using a computer code written especially for this problem. We have studied azimuthally symmetric beams with different rms-matched initial distributions. At each time step, the particles were given transverse deflections based on both the external- and self-forces. The radial self-forces were calculated from Gauss' law, assuming infinitely long cylindrical charge distributions. The objectives were to study the evolution of the beam distribution, U/w_0 and ϵ for different initial distributions and different tune ratios ω_1/ω_0 . We show two examples that illustrate the main results.

First, we present results for a space-charge-dominated beam with initial Gaussian distributions, both in position and divergence, truncated at two standard deviations and with an initial tune ratio $\omega_1/\omega_0 = 0.02$. We have introduced a radial distribution parameter, defined as $\rho = 1 - r/r_U$, where r is the average radius and r_U is the average radius for the equivalent (same second moments) uniform beam. For simple distributions, we find that ρ is positive for a peaked charge distribution, zero for a uniform beam, and negative for a hollow beam. However, the parameter ρ must be interpreted with caution, because in some cases the distributions are too complicated for such a simple characterization. Figures 1a and 1b show ρ and U/w_0 as a function of distance z/λ_p along the beamline, where a plasma length, $\lambda_p = (2\pi\epsilon_0 X^2/K\gamma^2)^{1/2}$, is the distance the beam travels during one plasma period. The parameter ρ indicates that the beam distribution undergoes damped radial oscillations between peaked and hollow configurations at the plasma frequency, a result that is confirmed by a more detailed examination of the charge distributions. The quantity U/w_0 is maximum at both the extreme peaked and hollow configurations, and therefore oscillates at twice the frequency of the parameter ρ . The minimum value of U/w_0 during these oscillations generally is not the zero value expected for an exactly uniform charge distribution. Examination of the beam cross section (not shown) when U/w_0 is minimum reveals the formation of a predominantly uniform beam, plus a low-density halo. We believe that two effects contribute to the nonzero minimum values of U/w_0 , the low-density halo, and a numerical error caused by statistical fluctuations in the charge distribution. Statistical fluctuations also appear to contribute to some of the details of the curves in Fig. 1, beyond about five plasma oscillations. Figure 1c shows the emittance ratio ϵ/ϵ_1 as a function of z/λ_p . Two overlapping curves are shown, one obtained from Eq. (3) with an evaluation of second moments, and the other from evaluation of U and using Eq. (13). The two curves are in excellent agreement and show a rapid initial growth that occurs during the first quarter of the first plasma period. A closer examination shows that the two curves are not in exact agreement, because of a slight mismatch caused by the emittance growth, an effect that is not included in the derivation of Eq. (13).

A second numerical example is shown in Fig. 2 for a beam with an initial semi-Gaussian or thermal distribution, corresponding to uniform charge density, Gaussian distribution in velocity space (truncated at four standard deviations), and with an initial tune ratio $\omega_1/\omega_0 = 0.25$. Figures 2a and 2b show ρ and U/w_0 versus z/λ_p . The parameter ρ rises from its initial value of zero to remain positive throughout, indicating a peaked distribution. An examination of the charge distribution reveals that the initial hard-edged beam evolved rapidly to one with a tail or soft edge. The quantity U/w_0 increases, which would imply

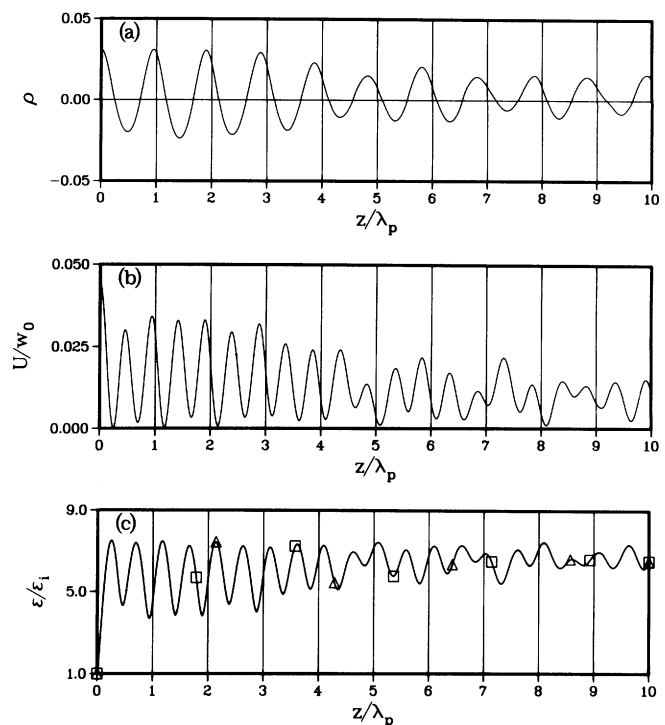


Fig. 1. Results of numerical simulation for an initial Gaussian charge distribution, truncated at two standard deviations, and initial tune ratio $\omega_1/\omega_0 = 0.02$. The abscissa is the distance z/λ_p along the beamline, where λ_p is the distance the beam travels in one plasma period. (a) Dimensionless radial distribution parameter ρ , defined in the text, is positive for a peaked charge distribution, zero for a uniform beam, and negative for a hollow beam. (b) Dimensionless nonlinear field energy U/w_0 . (c) Emittance ratio ϵ/ϵ_1 . There are two overlapping curves. The square symbols correspond to emittance calculated from second moments at each step, using Eq. (3). The triangles correspond to emittance calculated from nonlinear field energy at each step, using Eq. (13).

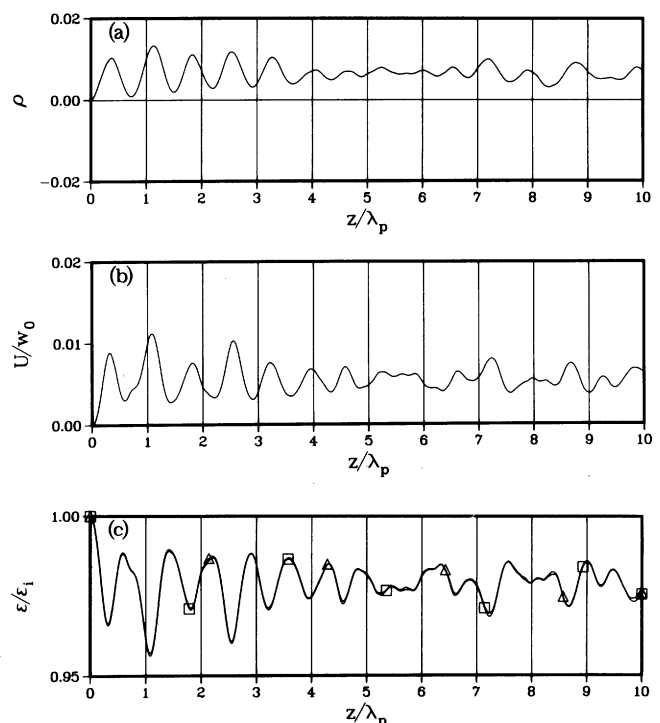


Fig. 2. Results of numerical simulation for an initial semi-Gaussian or thermal distribution, truncated at four standard deviations in velocity space, and initial tune ratio $\omega_1/\omega_0 = 0.25$. The quantities plotted are the same as in Fig. 1.

an emittance decrease according to Eq. (13). Figure 2c shows the rms emittance ratio ϵ_f/ϵ_i . Again, the two overlapping curves, one from Eq. (3) and the other from Eq. (13), are in excellent agreement, and both show a small decrease in the rms emittance. Thus, we conclude that not only can the rms emittance increase as the beam becomes more uniform, but also an rms emittance decrease is possible when the beam becomes less uniform. As in the previous example, damped oscillations are observed, but at present, we have no quantitative explanation of oscillation frequencies for beams that are not highly space-charge dominated.

Numerical studies have been made for other initial distributions and tune ratios and for distances up to 100 plasma lengths. Our interpretation of these results is that the initially rms-matched beam undergoes damped radial oscillations and evolves to a final state with a central core and sometimes with a low-density halo that contains a few per cent of the beam. The final charge density and final U/w_0 depend on the initial phase-space distribution and on the initial tune ratio. Therefore, Eq. (13) implies that the rms emittance growth ratio for a given initial phase-space distribution is a function only of the initial tune ratio.

Equations (10) and (13) describe rms emittance variation caused by any change in U/w_0 (associated with a change in the shape of the charge distribution) regardless of the detailed mechanism. In particular, unstable oscillation modes of the charge density, which are a well known property of the Kapchinskii-Vladimirskii (K-V) distribution below certain tune-ratio thresholds,¹⁵ can cause such changes in charge density. We have seen evidence for such instabilities in our numerical studies for an initial K-V beam. In such cases, we find that the charge density changes from the initial K-V uniform density to a nonuniform, slightly peaked configuration. As in the semi-Gaussian example described above, we observe an increase in U/w_0 and a small decrease in rms emittance.

Final, Uniform Charge-Density Approximation

In spite of the excellent agreement of Eq. (13) with the numerical simulation results, we are unable to predict the final rms emittance growth unless the final value of U/w_0 is known. In the extreme space-charge limit when ω_i and ϵ_i approach zero, we expect that the beam will evolve towards a final stationary state with a uniform charge distribution to obtain complete shielding of the linear applied focusing force. As an approximation for all initial tune ratios, we will assume that in the final state, $U/w_0 = 0$, which corresponds to a uniform beam with no halo. This approximation should be good in the space-charge-dominated limit, where the emittance growth is largest; in the emittance-dominated limit, where the approximation could lead to a large error in emittance growth, the emittance growth itself is small. In all cases, the approximation will result in an overestimate of the emittance growth.

With this approximation, the final emittance for an rms-matched beam can be written directly from Eq. (13) as

$$\frac{\epsilon_f}{\epsilon_i} \cong \left[1 + \frac{U_i}{2w_0} \left(\frac{\omega_0^2}{\omega_i^2} - 1 \right) \right]^{1/2}. \quad (14)$$

This equation corresponds to the upper limit emittance growth formula, derived by Struckmeier, Klabunde, and Reiser,⁸ if betatron frequencies are replaced by phase advances per quadrupole focusing period. We note that Eq. (14) depends on the properties of the initial beam, which can be easily calculated.

For a space-charge-dominated beam, Eq. (14) can be written as

$$\epsilon_f^2 \approx \epsilon_i^2 + \frac{1}{2} \left(\frac{Kv}{\omega_0} \right)^2 \left(\frac{U_i}{w_0} \right). \quad (15)$$

This result implies that as ϵ_i approaches zero, the final emittance approaches a minimum value that decreases with increased focusing force (larger ω_0), increases with perveance K , and increases with initial nonuniformity as measured by U_i/w_0 . This formula provides a quantitative description of the familiar lower limit on final emittance similar to that first reported in numerical studies by Chasman¹ for linear accelerator beams.

Numerical results have been used to test Eqs. (14) and (15). As an example, Fig. 3 shows ϵ_f/ϵ_i versus ω_i/ω_0 for an initial Gaussian distribution, truncated at four standard deviations. Values from the particle simulations are plotted and are in good agreement with the curve from Eq. (14).

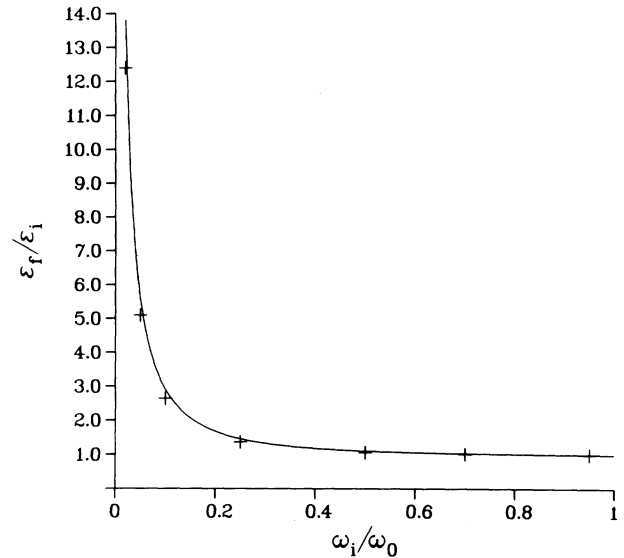


Fig. 3. Final emittance ratio versus initial tune ratio for an initial Gaussian distribution, truncated at four standard deviations. The curve is generated from the approximate formula, Eq. (14), and the plus symbols show the results of the particle simulations after 100 plasma periods.

Therefore, we conclude that emittance growth can be approximately predicted in advance from a knowledge only of the initial tune ratio and the initial value of U/w_0 . A comparison of Eq. (15) with numerical particle simulations is shown in Fig. 4, where ϵ_f versus ϵ_i is plotted for the same initial truncated Gaussian distribution at a value of $Kv/\omega_0 = 5.0 \times 10^{-5}$ m. The results from the simulation and the curve from Eq. (15) are, again, in close agreement.

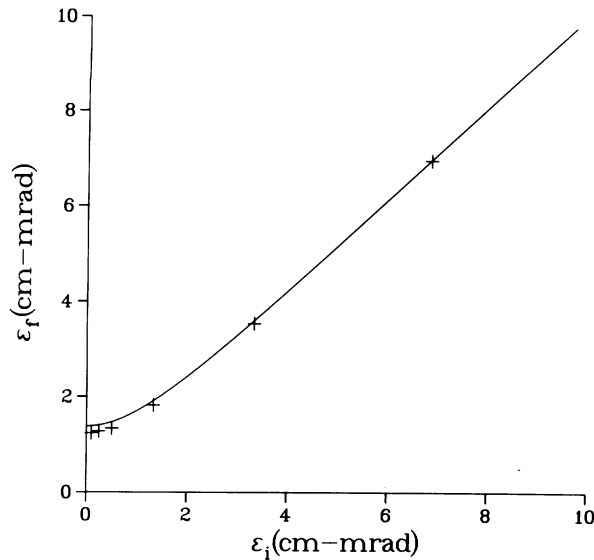


Fig. 4. Final emittance versus initial emittance for an initial Gaussian distribution truncated at four standard deviations. The curve is generated from the approximate formula, Eq. (15), for a value $Kv/\omega_0 = 5.0 \times 10^{-5}$ m, and the plus symbols show the results of particle simulations after 100 plasma periods.

Conclusions

We have attempted to evaluate the ideas expressed by Struckmeier, Klabunde, and Reiser⁸ that emittance growth is associated with the conversion of field energy to particle energy. We find that transverse energy conservation is valid for continuous channels, using the approximations given in this paper. We have obtained a formula, Eq. (10), which shows that rms emittance changes can be related to three separate factors: rms beam size, perveance, and changes in a quantity we call the nonlinear field energy, a quantity that depends on the shape of the charge distribution and corresponds to the residual space-charge electric-field energy of beams with nonuniform charge distributions. When integrated for the case of an rms-matched beam, we obtained Eq. (13), which is in excellent agreement with our numerical simulation studies. Furthermore, we have seen that this approach has led to good approximate formulas, including one that describes the well-known effect of minimum output emittance with decreasing input emittance.

The numerical studies reported in this paper have shown that the beams evolve to final states that are composed of a central core and a small halo, and that the emittance growth occurs within about one-quarter of a plasma period for high-intensity beams. From the numerical studies, we have seen examples of both the charge-density homogenization effects described by Ref. 8, resulting in an rms emittance increase, and of the reverse process, where the charge distribution evolves to one that is less uniform, resulting in a slight rms emittance decrease. The homogenization effect is a characteristic of peaked initial distributions in a highly space-charge-dominated case, whereas initially uniform distributions will become more nonuniform in a less space-charge-dominated situation, leading to a small rms emittance decrease. Equations (10) and (13) imply that rms emittance growth can be minimized by minimizing any decrease in the nonlinear field energy U . This can be done by producing the beam with an initial uniform charge distribution.

The studies of Ref. 8 were made for continuous beams in periodic quadrupole channels, and the results described in this paper apply for continuous beams in continuous focusing channels. Because continuous focusing channels represent a smooth approximation representation of a periodic channel, we anticipate that our results will be approximately valid for periodic systems in a smooth approximation, if the betatron frequencies ω_0 and ω_1 are replaced by phase advances per focusing period σ_0 and σ_1 . We plan to conduct further numerical studies to test the formulas for periodic systems.

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