Strong and Weak Coupling Limit of U(1) Lattice Model in Fourier Basis

Afsaneh Kianfar

Institute for Research in Fundamental Sciences
Schools of Particles and Accelerator (IPM)

Sep 7,2022

Outline

Transfer-Matrix in Fourier Basis

gauge variables

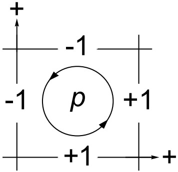
$$n_t : \quad \theta^l = a\, g\, A^{(\boldsymbol{r},i)}$$
$$n_t + 1 : \quad \theta'^l = a\, g\, A'^{(\boldsymbol{r},i)}$$

$$\langle n_t + 1 | \widehat{V} | n_t \rangle \;\propto\; e^{S_E(n_t, n_t+1)}$$

$$S_E(n_t, n_t+1) = -\frac{1}{2\,g^2} \sum_{\boldsymbol{r}} \sum_{i \neq j = 1}^{d} \Big[2 - \cos\left(\theta^{(\boldsymbol{r},i)} + \theta^{(\boldsymbol{r}+\widehat{i},j)} - \theta^{(\boldsymbol{r}+\widehat{j},i)} - \theta^{(\boldsymbol{r},j)}\right) - \cos\left(\theta'^{(\boldsymbol{r},i)} + \theta'^{(\boldsymbol{r}+\widehat{i},j)} - \theta'^{(\boldsymbol{r}+\widehat{j},i)} - \theta'^{(\boldsymbol{r},j)}\right) \Big] - \frac{1}{g^2} \sum_{\boldsymbol{r}} \sum_{i=1}^{d} \Big[1 - \cos\left(\theta^{(\boldsymbol{r},i)} - \theta'^{(\boldsymbol{r},i)}\right) \Big]$$

plaquette-link matrix M of dimension $N_{\mathrm{P}} \times N_{\mathrm{L}}$,

$$M^p_l = \begin{cases} \pm 1, & \text{link } l = (\boldsymbol{r}, \pm i) \text{ belongs to oriented plaquette } p \\ 0, & \text{otherwise.} \end{cases}$$



transfer-matrix $\widehat{V}$

$$\langle \boldsymbol{\theta}' | \widehat{V} | \boldsymbol{\theta} \rangle = \mathcal{A} \prod_p \exp\Big\{ -\frac{\gamma}{2} \Big[2 - \cos\left(M^p_{\ l}\, \theta^l \right) - \cos\left(M^p_{\ l}\, \theta'^l \right) \Big] \Big\} \times \prod_l \exp\Big\{ -\gamma \Big[1 - \cos\left(\theta^l - \theta'^l \right) \Big] \Big\}$$

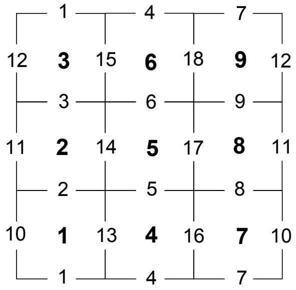
$$\langle \boldsymbol{k}' | \widehat{V} | \boldsymbol{k} \rangle = \mathcal{A}\, e^{-\gamma (N_{\mathrm{P}} + N_{\mathrm{L}})} \, (2\pi)^{N_{\mathrm{L}}} \sum_{\{n_p\}} \sum_{\{n'_p\}} \prod_p \mathrm{I}_{n_p}\Big(\frac{\gamma}{2} \Big) \, \mathrm{I}_{n'_p}\Big(\frac{\gamma}{2} \Big) \times \prod_l \mathrm{I}_{m_l}(\gamma)\, \delta\Big[(n_p + n'_p) M^p_{\ l} + k_l - k'_l \Big]$$

$$\boldsymbol{n}^{\mathbf{0}} \cdot \boldsymbol{M} = \mathbf{0}.$$

matrix M in the 2d lattice

$$M = \begin{pmatrix}
+ & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \\
0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 & 0 & 0 \\
- & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 & 0 \\
0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 \\
0 & 0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + & 0 \\
0 & 0 & 0 & - & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & + \\
0 & 0 & 0 & 0 & 0 & 0 & + & - & 0 & + & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & 0 & + & 0 & 0 & 0 & 0 & 0 & - & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0 & + & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & -
\end{pmatrix}$$

The numbering of links and plaquettes for 3×3 2d periodic lattice



matrix element in fourier basis

$$\langle \boldsymbol{k}'_{*\boldsymbol{q}'} | \widehat{V} | \boldsymbol{k}_{*\boldsymbol{q}} \rangle = \mathcal{A}\, e^{-\gamma(N_\mathrm{P}+N_\mathrm{L})} (2\pi)^{N_\mathrm{L}} \sum_{\{n_p^0\}} \sum_{\{n_p\}} \prod_p \mathrm{I}_{q_p - n_p}\left(\frac{\gamma}{2}\right) \mathrm{I}_{q'_p - n_p + n_p^0}\left(\frac{\gamma}{2}\right) \prod_l \mathrm{I}_{k_* + \sum_p n_p M^p_l}(\gamma)$$

Co-blocks

$$\boldsymbol{k}_{*\boldsymbol{q}} = \boldsymbol{k}_* + \boldsymbol{q} \cdot \boldsymbol{M}$$

$$e^{\mathrm{i} \sum_l k_l \theta^l} = e^{\mathrm{i}\, a\, g \sum_l k_l A^l} \to e^{\mathrm{i}\, g \int J \cdot A\, dx}$$

the matrix element $\langle \boldsymbol{k}' | \widehat{V} | \boldsymbol{k} \rangle$: is the transition amplitude between states with $\boldsymbol{k}$ and $\boldsymbol{k}'$ currents during the imaginary time interval 'a'.

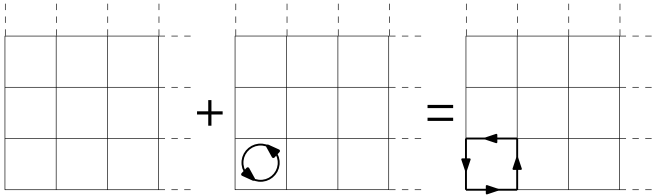
Currents, States and Blocks

A co-block of the vacuum state is found as the current-vector

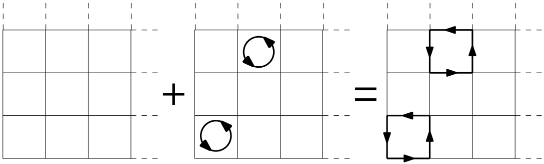
$$\boldsymbol{k}_{\mathbf{0;1}} = \boldsymbol{k}_{\mathbf{0}} + \boldsymbol{q}_{\mathbf{1}} \cdot \boldsymbol{M}$$

with

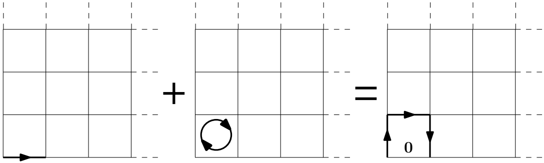
$$\boldsymbol{q}_{\mathbf{1}} = \underbrace{(1, 0, \cdots, 0)}_{N_{\mathrm{P}}}$$



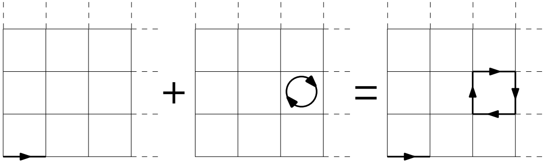
The graphical representation of $\boldsymbol{k}_{\mathbf{0;1}}$ as co-block of $\boldsymbol{k}_{\mathbf{0}} = \mathbf{0}$



Construction of a co-block of $\boldsymbol{k}_{\mathbf{0}}$ with two non-adjacent plaquette-currents.



$\boldsymbol{k}_{\mathbf{1};-\mathbf{1}} = \boldsymbol{k}_{\mathbf{1}} - \boldsymbol{q}_{\mathbf{1}} \cdot \boldsymbol{M}$ as a co-block of $\boldsymbol{k}_{\mathbf{1}}$ with 3 links having unit current.



A co-block of $\boldsymbol{k}_{\mathbf{1}}$ with five links having unit current.

Rules of Current Expansion in Strong Coupling

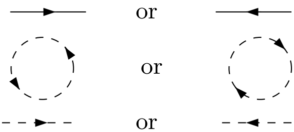
the ground-state is **unique** and belongs to the **$\boldsymbol{k_0}$'s block**. Energy and $\widehat{V}$ eigenvalues are related as

$$E_i = -\frac{1}{a} \ln v_i$$

Normalized transfer-matrix

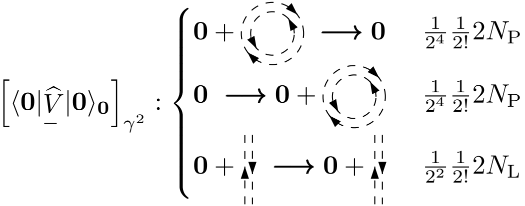
$$\widehat{V} = \mathcal{A}\, e^{-\gamma(N_{\rm L}+N_{\rm P})}(2\pi)^{N_{\rm L}}\ \underline{\widehat{V}}$$

$$\left[\langle \boldsymbol{k'} | \underline{\widehat{V}} | \boldsymbol{k} \rangle\right]_{m,m',\ell} = \mathcal{K}_{m,m',\ell}\ \frac{1}{2^{2m+2m'+\ell}} \frac{1}{m!\, m'!\, \ell!}\, \gamma^{m+m'+\ell}$$

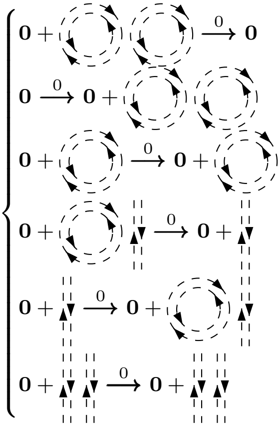


Current Expansion at Large Coupling

As examples of the **diagrammatic representation** of the terms in the strong coupling expansion, the diagrams contributing to the **vacuum to vacuum (v.t.v.)** transition at order γ^2 are



$$\left[\langle \mathbf{0}|\underline{\widehat{V}}|\mathbf{0}\rangle_{\mathbf{0}}\right]_{\gamma^4} :$$



$$\frac{1}{2^8}\,\frac{1}{4!}\,C_4^2\left(2N_\mathrm{P}^2 - N_\mathrm{P}\right)$$

$$\frac{1}{2^8}\,\frac{1}{4!}\,C_4^2\left(2N_\mathrm{P}^2 - N_\mathrm{P}\right)$$

$$\frac{1}{2^8}\,\frac{1}{2!2!}\,C_2^1 C_2^1 N_\mathrm{P}^2$$

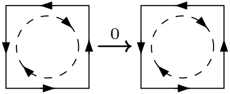
$$\frac{1}{2^6}\,\frac{1}{2!2!}\,C_2^1 C_2^1 N_\mathrm{P} N_\mathrm{L}$$

$$\frac{1}{2^6}\,\frac{1}{2!2!}\,C_2^1 C_2^1 N_\mathrm{P} N_\mathrm{L}$$

$$\frac{1}{2^4}\,\frac{1}{4!}\,C_4^2\left(2N_\mathrm{L}^2 - N_\mathrm{L}\right)$$

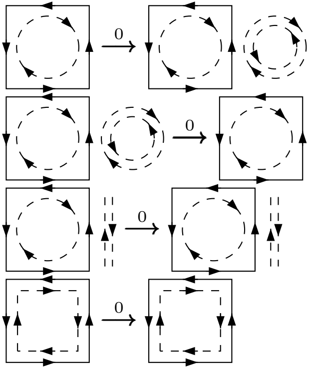
$$\langle \mathbf{0}|\underline{\widehat{V}}|\mathbf{0}\rangle_{\mathbf{0}} = 1 + \left(\frac{N_\mathrm{P}}{8} + \frac{N_\mathrm{L}}{4}\right)\gamma^2 + \left(-\frac{N_\mathrm{L}}{64} + \frac{N_\mathrm{L}^2}{32} - \frac{N_\mathrm{P}}{512} + \frac{N_\mathrm{P} N_\mathrm{L}}{32} + \frac{N_\mathrm{P}^2}{128}\right)\gamma^4 + \mathrm{O}(\gamma^6)$$

$$\left[\langle \mathbf{1}|\underline{\widehat{V}}|\mathbf{1}\rangle_{\mathbf{0}}\right]_{\gamma^2} :$$



$$\frac{\gamma^2}{16}$$

$$\left[\langle \mathbf{1}|\underline{\widehat{V}}|\mathbf{1}\rangle_{\mathbf{0}}\right]_{\gamma^4} :$$



$$\frac{1}{2^8}\,\frac{1}{3!}\,C_3^1\,(2N_{\mathrm{P}}-1)$$

$$\frac{1}{2^8}\,\frac{1}{3!}\,C_3^1\,(2N_{\mathrm{P}}-1)$$

$$\frac{1}{2^6}\,\frac{1}{2!}\,C_2^1\,N_{\mathrm{L}}$$

$$\frac{1}{2^4}\,\frac{1}{4!}\,C_4^2\,4$$

$$\langle \mathbf{1}|\underline{\widehat{V}}|\mathbf{1}\rangle_{\mathbf{0}} = \frac{\gamma^2}{16} + \left(\frac{15}{256} + \frac{N_{\mathrm{L}}}{64} + \frac{N_{\mathrm{P}}}{128}\right)\gamma^4 + \cdots$$

$$\langle \mathbf{1} | \underline{\widehat{V}} | \mathbf{0} \rangle_{\mathbf{0}} = \frac{\gamma}{4} + \left(\frac{-1}{128} + \frac{N_\mathrm{P}}{32} + \frac{N_\mathrm{L}}{16} \right) \gamma^3 + \left(\frac{49}{3072} - \frac{3N_\mathrm{L}}{512} + \frac{N_\mathrm{L}^2}{128} - \frac{3N_\mathrm{P}}{2048} + \frac{N_\mathrm{L} N_\mathrm{P}}{128} + \frac{N_\mathrm{P}^2}{512} \right) \gamma^5 + \cdots$$

$$\langle \boldsymbol{k'}_{*\boldsymbol{q'}} | \underline{\widehat{V}} | \boldsymbol{k}_{*\boldsymbol{q}} \rangle_{\boldsymbol{k}_*} = \gamma^h \left(c_0 + c_2\, \gamma^2 + c_4\, \gamma^4 + c_6\, \gamma^6 + \cdots \right)$$

$$|\boldsymbol{k}_*| = \sum_{l=1}^{N_\mathrm{L}} |k_{*l}|$$

$$h = |\boldsymbol{k}_*| + |\boldsymbol{q}| + |\boldsymbol{q'}|$$

$$|\boldsymbol{q}| = \sum_{p=1}^{N_\mathrm{P}} |q_p|, \qquad |\boldsymbol{q'}| = \sum_{p=1}^{N_\mathrm{P}} |q'_p|$$

Spectrum in Strong Coupling Limit

$$\gamma = 0: \; v_0^{(0)} = 1, \qquad v_{q\neq 0}^{(0)} = 0$$

$\varepsilon_0^{(0)} = 0$ for the ground-state and $\varepsilon_{q\neq 0}^{(0)} \to +\infty$ for all other states.

$$\underline{\widehat{V}} = \widehat{V}^0 + \bar{V}$$

eigenvalue in vacuum block

$$\begin{aligned} v_0 &= 1 + \bar{V}_{00} + 2N_{\mathrm{P}} \bar{V}_{0,\pm 1}^2 \\ &= 1 + \frac{1}{4}(N_{\mathrm{P}} + N_{\mathrm{L}})\gamma^2 + \mathrm{O}(\gamma^4) \end{aligned}$$

eigenvector

$$\begin{aligned} \vec{v}_0 =& (1, 0, 0, \cdots 0) + \sum_{|q|=1} \bar{V}_{0,q} \; \vec{q} \\ =& \Big(1, \underbrace{\frac{\gamma}{4}, \frac{\gamma}{4}, \cdots, \frac{\gamma}{4}}_{2N_{\mathrm{P}}}\Big) + \mathrm{O}(\gamma^2) \end{aligned}$$

$$v_0 = 1 + \frac{1}{4}(N_\mathrm{P} + N_\mathrm{L})\gamma^2 + \left(\frac{-N_\mathrm{L}}{64} + \frac{N_\mathrm{L}^2}{32} - \frac{N_\mathrm{P}}{64} + \frac{N_\mathrm{L} N_\mathrm{P}}{16} + \frac{N_\mathrm{P}^2}{32} \right) \gamma^4 + \mathrm{O}(\gamma^6)$$

eigenvalue in $\boldsymbol{k_1}$ block

$$\begin{aligned} v_0^{\boldsymbol{k_1}} =& \frac{\gamma}{2} + \left(\frac{-1}{16} + \frac{N_\mathrm{L}}{8} + \frac{N_\mathrm{P}}{8} \right) \gamma^3 \\ &+ \left(\frac{7}{384} - \frac{3N_\mathrm{L}}{128} + \frac{N_\mathrm{L}^2}{64} - \frac{3N_\mathrm{P}}{128} + \frac{N_\mathrm{L} N_\mathrm{P}}{32} + \frac{N_\mathrm{P}^2}{64} \right) \gamma^5 + \mathrm{O}(\gamma^7) \end{aligned}$$

eigenvector in $\boldsymbol{k_1}$ block

$$\begin{aligned} \vec{v}_0^{\boldsymbol{k_1}} =& (1, 0, 0, \cdots 0) + \frac{1}{\gamma/2} \sum_{|q|=1} \bar{V}_{0,q}^{\boldsymbol{k_1}} \ \vec{q} \\ =& \Big(1, \underbrace{\frac{\gamma}{4}, \frac{\gamma}{4}, \cdots, \frac{\gamma}{4}}_{2N_\mathrm{P}}\Big) + \mathrm{O}(\gamma^2) \end{aligned}$$

 Spectrum in Strong Coupling Limit

$$v_0^{\boldsymbol{k_2}} = \frac{\gamma^2}{8} + \left(-\frac{1}{48} + \frac{N_\mathrm{L}}{32} + \frac{N_\mathrm{P}}{32} \right) \gamma^4 + \left(\frac{23}{3072} - \frac{11 N_\mathrm{L}}{1536} + \frac{N_\mathrm{L}^2}{256} - \frac{11 N_\mathrm{P}}{1536} + \frac{N_\mathrm{L} N_\mathrm{P}}{128} + \frac{N_\mathrm{P}^2}{256} \right) \gamma^6 + \mathrm{O}(\gamma^8)$$

$$v_0^{\boldsymbol{k_{11'}}} = \frac{\gamma^2}{4} + \left(-\frac{1}{16} + \frac{N_\mathrm{L}}{16} + \frac{N_\mathrm{P}}{16} \right) \gamma^4 + \left(\frac{7}{384} - \frac{5 N_\mathrm{L}}{256} + \frac{N_\mathrm{L}^2}{128} - \frac{5 N_\mathrm{P}}{256} + \frac{N_\mathrm{L} N_\mathrm{P}}{64} + \frac{N_\mathrm{P}^2}{128} \right) \gamma^6 + \mathrm{O}(\gamma^8)$$

$$v_0^{\boldsymbol{k_3}} = \frac{\gamma^3}{48} + \left(-\frac{1}{256} + \frac{N_\mathrm{L}}{192} + \frac{N_\mathrm{P}}{192} \right) \gamma^5 + \left(\frac{17}{10240} - \frac{N_\mathrm{L}}{768} + \frac{N_\mathrm{L}^2}{1536} - \frac{N_\mathrm{P}}{768} + \frac{N_\mathrm{L} N_\mathrm{P}}{768} + \frac{N_\mathrm{P}^2}{1536} \right) \gamma^7 + \mathrm{O}(\gamma^9)$$

$$Z[J] = \frac{\int \mathscr{D}\phi \, \exp\left[\mathrm{i} \int dx \, (\mathscr{L} + J\,\phi)\right]}{\int \mathscr{D}\phi \, \exp\left[\mathrm{i} \int dx \, \mathscr{L}\right]}$$

$$v_i/v_0 = \exp\left[- a(\varepsilon_i - \varepsilon_0)\right]$$

$$\frac{v_0^{\boldsymbol{k_1}}}{v_0} = \frac{\gamma}{2}\left(1 - \frac{\gamma^2}{8} + \frac{7\gamma^4}{192} + \cdots\right)$$

$$\frac{v_0^{\boldsymbol{k_2}}}{v_0} = \frac{\gamma^2}{8}\left(1 - \frac{\gamma^2}{6} + \frac{23\gamma^4}{384} + \cdots\right)$$

$$\frac{v_0^{\boldsymbol{k_{11'}}}}{v_0} = \frac{\gamma^2}{4}\left(1 - \frac{\gamma^2}{4} + \frac{7\gamma^4}{96} + \cdots\right)$$

$$\frac{v_0^{\boldsymbol{k_3}}}{v_0} = \frac{\gamma^3}{48}\left(1 - \frac{3\gamma^2}{16} + \frac{51\gamma^4}{640} + \cdots\right)$$

Phys. Rev. D 104 (2021) 094506, 2107.07877

Gauss Law Constraint in Fourier Basis

wave-function

$$\psi_{\mathrm{C}}[\boldsymbol{\theta}] = \exp\Big[\mathrm{i} \sum_{l \in \mathrm{C}} J_l \, \theta^l \Big]$$

$$\widetilde{\psi}_{\mathrm{C}}[\boldsymbol{k}] = \frac{1}{(2\pi)^{N_{\mathrm{L}}/2}} \int_{-\pi}^{\pi} \prod_{l} \mathrm{d}\theta_l \; \exp\Big[\mathrm{i}\Big(\sum_{l \in \mathrm{C}} J_l \, \theta^l - \boldsymbol{k} \cdot \boldsymbol{\theta} \Big)\Big]$$

$$\widetilde{\psi}_{\mathrm{C}}[\boldsymbol{k}] = (2\pi)^{N_{\mathrm{L}}/2} \prod_{l \in \mathrm{C}} \delta(k_l - J_l) \prod_{l' \notin \mathrm{C}} \delta(k_{l'})$$

 Gauss Law Constraint in Fourier Basis

For the case of **the 1d periodic spatial lattice**, the only closed loop is the one around the entire the periodic spatial direction.

$$\langle k|\widehat{V}|k\rangle = \mathcal{A}\left(e^{-\gamma}\,\mathrm{I}_k(\gamma)\right)^{N_\mathrm{L}}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

the exact spectrum of 1d model is found:

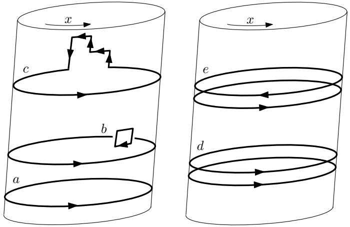
$$E_k = -\frac{1}{a}\ln\langle k|\widehat{V}|k\rangle$$

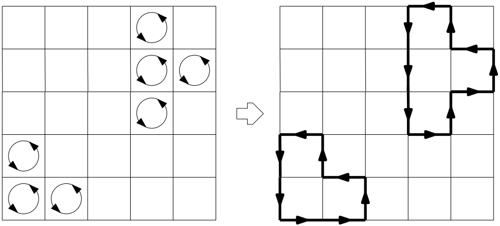
$$\mathrm{I}_k(\gamma) \simeq \frac{e^{\gamma}}{\sqrt{2\pi\gamma}}\, e^{-(k^2-1/4)/(2\gamma)} \left[1 + \mathrm{O}\left(\frac{1}{\gamma^2}\right)\right], \qquad \gamma \gg k$$

the continuum energy density (energy per link) expected by the classical model is recovered

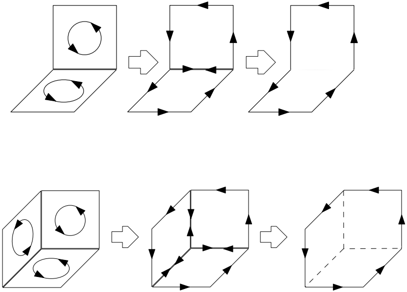
$$\frac{E_k}{N_\mathrm{L}} \simeq \frac{(k^2 - {}^1\!/\!_4)g^2}{2\,a}, \qquad g \ll 1, \quad k \in \mathbb{Z}.$$

Left: periodic loop current a, and two of its co-blocks b and c. **Right:** d as a two spaced equal periodic loop currents, and e as a two spaced opposite periodic loop currents. In fact, e belongs to the vacuum block.

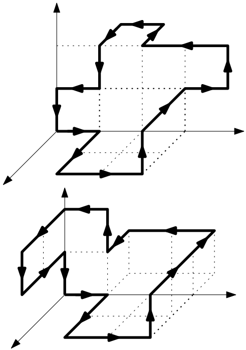




Examples of 2d gauge invariant closed currents from the vacuum block.



Examples of 3d closed edge currents from the vacuum block.



General 3d closed currents from the vacuum block.

Transfer-Matrix in Weak Coupling Limit

In the weak coupling limit $g \ll 1$ the configurations with $\theta \ll 1$ find the dominant contribution.for following vector with $2N_{\mathrm{L}}$ components

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{A}' \end{pmatrix}$$

the elements of the transfer-matrix are given by means of the following quadratic form

$$\langle \boldsymbol{\theta'} | \widehat{V} | \boldsymbol{\theta} \rangle = \mathcal{A} \exp\left(-\frac{a^2}{2} \boldsymbol{\eta}^T \boldsymbol{C} \, \boldsymbol{\eta} + \mathrm{O}(g^2) \right)$$

matrix $\boldsymbol{C}$ has $2N_{\mathrm{L}} \times 2N_{\mathrm{L}}$ dimensions

$$\begin{aligned} \boldsymbol{C} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \mathbb{1}_{\mathrm{L}} + \frac{1}{2} \mathbb{1}_2 \otimes \boldsymbol{M}^T \boldsymbol{M} \\ &= \left(\begin{array}{c|c} \mathbb{1}_{\mathrm{L}} + \frac{1}{2} \boldsymbol{M}^T \boldsymbol{M} & -\mathbb{1}_{\mathrm{L}} \\ \hline -\mathbb{1}_{\mathrm{L}} & \mathbb{1}_{\mathrm{L}} + \frac{1}{2} \boldsymbol{M}^T \boldsymbol{M} \end{array} \right) \end{aligned}$$

 Transfer-Matrix in Weak Coupling Limit

$$\boldsymbol{\kappa} = \begin{pmatrix} \boldsymbol{k} \\ -\boldsymbol{k}' \end{pmatrix}$$

So in Fourier Basis

$$\langle \boldsymbol{k}' | \widehat{V} | \boldsymbol{k} \rangle \simeq \mathcal{A} \frac{g^{2N_\mathrm{L}}}{(2\pi)^{N_\mathrm{L}}} \int_{-\pi/g}^{\pi/g} \mathrm{d}\boldsymbol{\eta} \, \exp\left(-\frac{1}{2} \boldsymbol{\eta}^T \boldsymbol{C} \, \boldsymbol{\eta} + \mathrm{i} \, g \, \boldsymbol{\eta}^T \boldsymbol{\kappa} \right)$$

$$\tilde{\boldsymbol{C}} = \boldsymbol{P}^{-1} \boldsymbol{C} \boldsymbol{P}$$
$$\tilde{\boldsymbol{\eta}} = \boldsymbol{P}^{-1} \boldsymbol{\eta}$$
$$\tilde{\boldsymbol{\eta}}^T = \boldsymbol{\eta}^T \boldsymbol{P}$$
$$\tilde{\boldsymbol{\kappa}} = \boldsymbol{P}^{-1} \boldsymbol{\kappa}$$

$\tilde{\boldsymbol{C}}_\mathrm{u}$ is diagonal, and $\tilde{\boldsymbol{C}}_\mathrm{d} = \boldsymbol{0}$.

$$\tilde{\boldsymbol{C}} = \left(\begin{array}{c|c} \tilde{\boldsymbol{C}}_\mathrm{u} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \tilde{\boldsymbol{C}}_\mathrm{d} \end{array} \right)$$

	No. of sites	No. of links	dim. of $\tilde{C}_{\mathrm{d}}$
2d lattice	N_s^2	$N_{\mathrm{L}} = 2N_s^2$	$N_{\mathrm{d}} = N_s^2 + 1$
3d lattice	N_s^3	$N_{\mathrm{L}} = 3N_s^3$	$N_{\mathrm{d}} = N_s^3 + 2$

For periodic lattices in two and three dimensions the size of block $\tilde{C}_{\mathrm{d}}$ is given by explicit representations of M.

$$\langle \boldsymbol{k'} | \widehat{V} | \boldsymbol{k} \rangle = \mathcal{A} \frac{g^{2N_{\mathrm{L}}-N_{\mathrm{d}}}}{(2\pi)^{N_{\mathrm{L}}-N_{\mathrm{d}}}} \, \delta(\tilde{\boldsymbol{\kappa}}_{\mathrm{d}}) \int_{-\pi/g}^{\pi/g} \mathrm{d}\tilde{\boldsymbol{\eta}}_{\mathrm{u}} \, \exp\left(-\frac{1}{2} \tilde{\boldsymbol{\eta}}_{\mathrm{u}}^T \tilde{\boldsymbol{C}}_{\mathrm{u}} \, \tilde{\boldsymbol{\eta}}_{\mathrm{u}} + \mathrm{i} \, g \, \tilde{\boldsymbol{\eta}}_{\mathrm{u}}^T \tilde{\boldsymbol{\kappa}}_{\mathrm{u}} \right)$$

$$\langle \boldsymbol{k'}_{*\boldsymbol{q'}} | \widehat{V} | \boldsymbol{k}_{*\boldsymbol{q}} \rangle = \mathcal{A} \frac{g^{2N_{\mathrm{L}}-N_{\mathrm{d}}}}{(2\pi)^{N_{\mathrm{L}}-N_{\mathrm{d}}}} \sqrt{\frac{(2\pi)^{2N_{\mathrm{L}}-N_{\mathrm{d}}}}{\det \tilde{\boldsymbol{C}}_{\mathrm{u}}}} \left[\exp\left(-\frac{g^2}{2} \tilde{\boldsymbol{\kappa}}_{\mathrm{u}}^T \tilde{\boldsymbol{C}}_{\mathrm{u}}^{-1} \tilde{\boldsymbol{\kappa}}_{\mathrm{u}} \right) + \mathrm{O}\left(g \, e^{-\pi^2/g^2} \right) \right]$$

$$g \ll 1$$

Spectrum in Weak Coupling

$$\mathbb{P} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathbb{1}_{\mathrm{L}} & \mathbb{1}_{\mathrm{L}} \\ \hline -\mathbb{1}_{\mathrm{L}} & \mathbb{1}_{\mathrm{L}} \end{array} \right)$$

the matrix $\boldsymbol{C}$ comes to the form

$$\boldsymbol{C'} = \mathbb{P}^{-1} \boldsymbol{C}\, \mathbb{P} = \left(\begin{array}{c|c} 2\, \mathbb{1}_{\mathrm{L}} + \frac{1}{2} \boldsymbol{M}^T \boldsymbol{M} & 0 \\ \hline 0 & \frac{1}{2} \boldsymbol{M}^T \boldsymbol{M} \end{array} \right)$$

$$\langle \boldsymbol{A'} | \widehat{V} | \boldsymbol{A} \rangle = \mathcal{A} \exp\left(-\frac{1}{2}\, \boldsymbol{A}^{-} \left(\mathbb{1}_{\mathrm{L}} + \frac{1}{4} \boldsymbol{M}^T \boldsymbol{M} \right) \boldsymbol{A}^{-} - \frac{1}{2}\, \boldsymbol{A}^{+} \left(\frac{1}{4} \boldsymbol{M}^T \boldsymbol{M} \right) \boldsymbol{A}^{+} \right)$$

in which

$$\boldsymbol{A}^{\pm} = \boldsymbol{A} \pm \boldsymbol{A'}$$

$$\boldsymbol{M}^T\boldsymbol{M}\cdot\boldsymbol{\xi} = 4\xi^2\,\boldsymbol{\xi}$$

for which $\langle \boldsymbol{\xi}|\boldsymbol{\xi'}\rangle = \delta_{\xi\xi'}$.
In this basis, obviously the **zero eigenvalues** only contribute to the **A^- part**, and the matrix element finds the form

$$\begin{aligned}\langle \boldsymbol{A'}|\widehat{V}|\boldsymbol{A}\rangle = \mathcal{A}\,\exp\Bigg(-\frac{1}{2}\sum_{\{\xi=0\}}(A_\xi - A'_\xi)^2 \\ -\frac{1}{2}\sum_{\xi\neq 0}\Big((1+\xi^2)(A_\xi - A'_\xi)^2 + \xi^2(A_\xi + A'_\xi)^2\Big)\Bigg)\end{aligned}$$

$$\begin{aligned}\mathrm{i}\,g\,\boldsymbol{\eta}^T\boldsymbol{\kappa} &= \mathrm{i}\,g\,(\boldsymbol{A}\cdot\boldsymbol{k} - \boldsymbol{A'}\cdot\boldsymbol{k'}) \\ &= \frac{\mathrm{i}}{2}g\Big(\sum_{\{\xi=0\}}(A^+_\xi k^-_\xi - A^-_\xi k^+_\xi) + \sum_{\xi\neq 0}(A^+_\xi k^-_\xi - A^-_\xi k^+_\xi)\Big)\end{aligned}$$

$$\boldsymbol{k}^\pm = \boldsymbol{k} \pm \boldsymbol{k'}$$

the contribution of zero modes:

$$E_{\text{free}} = \sum_{\{\xi=0\}} \frac{1}{2}\left(\frac{g\,k^{+}_{\xi}}{2}\right)^2 = \sum_{\{\xi=0\}} \frac{g^2 k^2_{\xi}}{2}$$

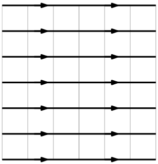
$$P_0 = \sum_{\{\xi=0\}} |\xi\rangle\langle\xi|$$

which satisfies $P_0^2 = P_0$. By the explicit form of the matrix $\boldsymbol{M}$, it is a simple task to check that the result of the projection of the mentioned allowed states to the subspace by zero modes is a **uniform electric flux.**

$$\boldsymbol{k} = (\underbrace{1, 1, \cdots, 1}_{N_s^2}, \underbrace{0, 0, \cdots, 0}_{N_s^2})^T$$

$$P_0\,(\boldsymbol{q}\cdot\boldsymbol{M})^T = \sum_{\{\xi=0\}} |\xi\rangle\underbrace{\langle\xi|\boldsymbol{M}}_{0}{}^T\cdot\boldsymbol{q}^T = 0$$

 Spectrum in Weak Coupling



The thick lines represent $\boldsymbol{k}$ as uniform electric fluxes on links of a 2d lattice.

The non-zero modes represent a harmonic oscillator dynamics

$$\langle x'|\hat{V}|x\rangle = \sqrt{\frac{M}{2\pi}}\exp\left(-\frac{1}{2}M(x-x')^2 - \frac{1}{2}M\omega^2\left(\frac{x+x'}{2}\right)^2\right)$$

with the spectrum $E_r = (r + \frac{1}{2})\,\omega$ with $r = 0, 1, 2, \cdots$.

$$\omega_\xi^2 = \frac{4\xi^2}{1+\xi^2}$$

$$E_{\text{tot}} = \sum_{\{\xi=0\}} \frac{g^2 k_\xi^2}{2\,a} + \frac{1}{a}\sum_{\xi\neq 0}\left(r_\xi + \frac{1}{2}\right)\omega_\xi$$

 Spectrum in Weak Coupling

$$4\xi_{m,n}^2 = 4\left(\sin^2\frac{\pi m}{N_s} + \sin^2\frac{\pi n}{N_s}\right)$$

$$\omega_{m,n}^2 \simeq 4\xi_{m,n}^2 = \frac{4a^2\pi^2}{L^2}(m^2+n^2), \qquad m,n \ll N_s$$

$$E_{\text{tot}} = \sum_{\{\xi=0\}} \frac{g^2 k_\xi^2}{2\,a} + \frac{2\pi}{L}\sum_{m,n}\left(r_{m,n}+\frac{1}{2}\right)\sqrt{m^2+n^2}$$

Thanks For Your Attention