Stochastic Effects in Axion Inflation

Alireza Talebian

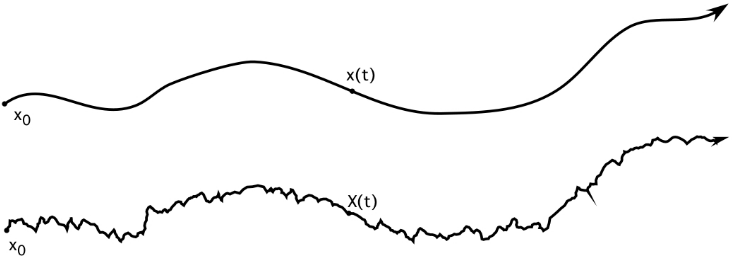
School of Astronomy

General Weekly Meeting of the School of Particles&Accelerators

Oct 12, 2022

Outline

1. Stochastic Differential Equations
   - Examples
   - Fokker-Planck Equation
2. Stochastic Processes in Early Universe
   - Inflation
   - EM fields during Inflation
   - Primordial Magnetic Fields
   - Primordial Helical Magnetic fields
   - Stochastic analysis of Axion inflation
3. Summary and Conclusion



stochastic process $X(\cdot)$: solution of SDE

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t))\, \xi(t)\,, & (t > 0) \\ X(0) = X_0, \end{cases}$$

White noise:

$$\langle \xi(t)\ \xi(t') \rangle = \delta(t - t')\,.$$

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t))\,\xi(t)\,, & (t > 0) \\ X(0) = X_0, \end{cases}$$

White noise:

$$\langle \xi(t)\ \xi(t') \rangle = \delta(t - t')\,.$$

$$X(t) = X_0 + \int_0^t b(X, s)\ \mathrm{d}s + \int_0^t B(X, s)\ \mathrm{d}W$$

Wiener Process (Brownian motion):

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \xi(t)$$

$$\langle W(t) \rangle = 0\,, \quad \langle W(t)W(s) \rangle = min\{t, s\}\,, \quad \langle W^2(t) \rangle = t\,,$$

Examples of SDE

Examples

- Wiener process (Brownian motion)
- Stock prices (geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process (mean-reverting process)

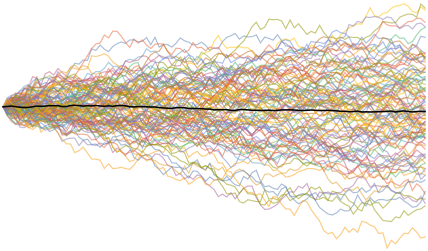


Figure: Wiener

$$dX_t = \sigma \, dW_t$$

σ : *constant.*

Examples of SDE

Examples

- Wiener process (Brownian motion)
- Stock prices (geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process (mean-reverting process)

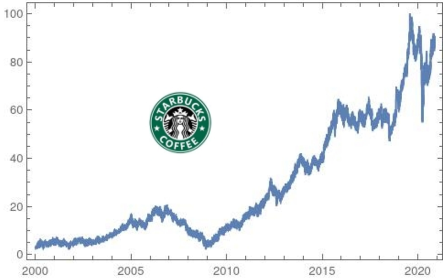


Figure: Starbucks Corporation (SBUX) Stock Price

$$\begin{cases} \mathrm{d}S_t = \mu\ S_t\ \mathrm{d}t + \sigma\ S_t\ \mathrm{d}W_t\ , \\ S(0) = s_0, \end{cases}$$

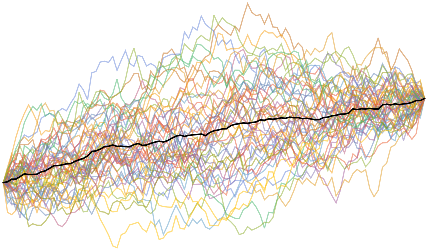
$$S(t) = s_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t}$$

$\mu > 0$: Drift , σ: volatility .

Examples of SDE

Examples

- Wiener process (Brownian motion)
- Stock prices (geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process (mean-reverting process)



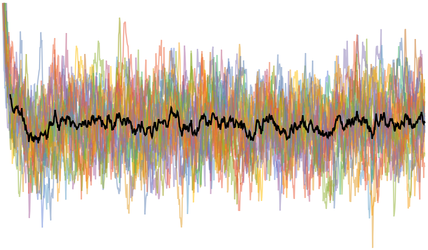
$$\begin{cases} \mathrm{d}B_{\mathrm{t}} = -\dfrac{B_{\mathrm{t}}}{1-t}\,\mathrm{d}t + \mathrm{d}W_{\mathrm{t}}\,, \\ B(0) = 0, \end{cases}$$

$$B(t) = (1-t)\int_0^t \frac{1}{1-s}\mathrm{d}W_{\mathrm{s}}$$

Examples of SDE

Examples

- Wiener process (Brownian motion)
- Stock prices (geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process (mean-reverting process)



$$\begin{cases} \mathrm{d}X_{\mathrm{t}} = -b\ X_{\mathrm{t}}\ \mathrm{d}t + \sigma\ \mathrm{d}W_{\mathrm{t}}\,, \\ X(0) = X_0, \end{cases}$$

$$X(t) = X_0\ e^{-b\,t} + \sigma \int_0^t e^{-b(t-s)}\ \mathrm{d}W$$

Friction $b > 0\,,$ diffusion: σ

Fokker-Planck Equation

Probability density

$f_X(x,t)$: probability of X falling within the infinitesimal interval $[x, x+\mathrm{d}x]$ at the moment t

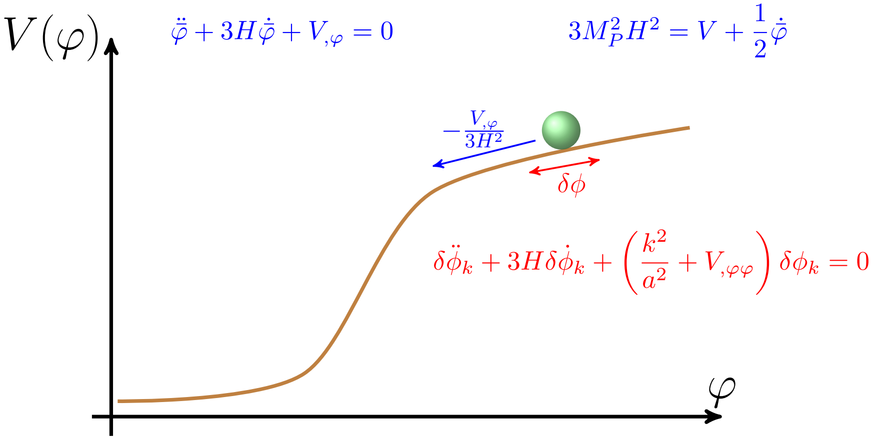
$$\mathrm{d}X_\mathrm{t} = \mu(X_\mathrm{t}, t)\ \mathrm{d}t + \sigma(X_\mathrm{t}, t)\ \mathrm{d}W_\mathrm{t}$$

$$\frac{\partial}{\partial t} f_X(x,t) = -\frac{\partial}{\partial x}\left(\mu(x,t) f_X(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(D(x,t) f_X(x,t)\right)$$

drift $\mu(X_\mathrm{t}, t)$ and diffusion coefficient $D(X_\mathrm{t}, t) = \sigma^2(X_\mathrm{t}, t)/2$

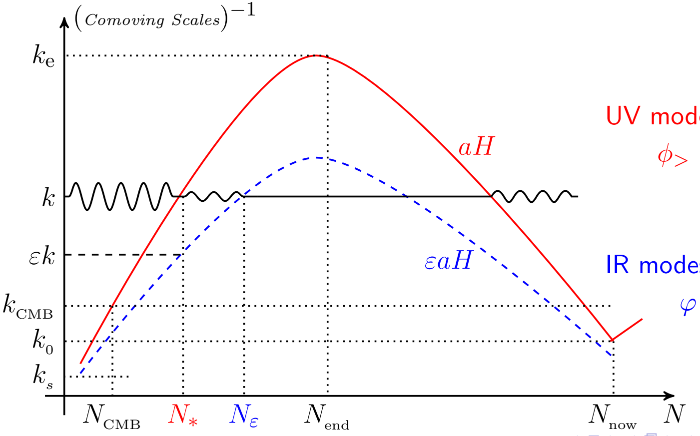
Inflation: formal approach

Inflaton (quantum) field: $\hat{\Phi} = \bar{\varphi} + \delta\phi$
Classical background (fixed) : $\bar{\varphi} \Rightarrow$ acceleration $\Rightarrow$ Homogeneous
Small quantum perturbations : $\delta\phi \Rightarrow$ Structure



$$\hat{\Phi} = \varphi + \phi_{>}$$

Coarse-graining: $W_H(k,t) = \Theta\left(k - \varepsilon aH\right)$



UV modes:

$\phi_{>}$: $k > \varepsilon aH$

IR modes:

φ : $k < \varepsilon aH$

Stochastic Inflation Langevin equation

Split the quantum fields ($\hat{\Phi}$, $\hat{\Pi}$) into long and short modes

$$\hat{\Phi} = \varphi + \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Theta\left(k - \varepsilon a(t) H\right)\, \hat{\phi}_{\mathbf{k}}(t) e^{i\mathbf{k}.\mathbf{x}}$$

$$\dot{\hat{\Phi}} = \hat{\Pi} = \pi + \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Theta\left(k - \varepsilon a(t) H\right)\, \dot{\hat{\phi}}_{\mathbf{k}}(t) e^{i\mathbf{k}.\mathbf{x}}$$

time-dependent window function

Langevin's equation
$$\begin{cases} \dfrac{d\varphi}{dN} = \pi + \hat{\xi}_\varphi \\ \dfrac{d\pi}{dN} = -(3 - \epsilon_H)\pi - \dfrac{V_{,\varphi}}{3H^2} + \hat{\xi}_\pi \end{cases}$$

e-folding number: $dN = H dt$

Stochastic inflation: Stochastic Noises

For a light scalar field, Qunatum stochastic noises:

$$\hat{\xi}_\varphi(t,\mathbf{x}) = -\frac{\mathrm{d}k_c}{\mathrm{d}t}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\ \delta(k-k_c)\ e^{i\boldsymbol{k}.\mathbf{x}}\ \hat{\phi}_{\mathbf{k}}(t)\,,$$

$$\hat{\xi}_\pi(t,\mathbf{x}) = -\frac{\mathrm{d}k_c}{\mathrm{d}t}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\ \delta(k-k_c)\ e^{i\boldsymbol{k}.\mathbf{x}}\ \dot{\hat{\phi}}_{\mathbf{k}}(t)\,, \qquad k_\mathrm{c} = \varepsilon a(t)H$$

commute as $\varepsilon \to 0$

$$\frac{\langle[\hat{\xi}_\varphi,\hat{\xi}_\pi]\rangle}{\langle\hat{\xi}_\varphi,\hat{\xi}_\varphi\rangle} \ll 1$$

and become classic noises:

$$\langle\hat{\xi}_\varphi(N_1)\ \hat{\xi}_\varphi(N_2)\rangle = \left(\frac{H}{2\pi}\right)^2 \delta(N_1-N_2)$$

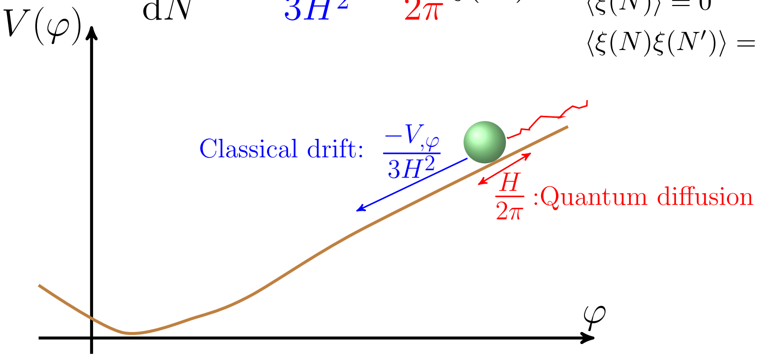
$$\langle\hat{\xi}_\pi(N_1)\ \hat{\xi}_\pi(N_2)\rangle \sim \mathcal{O}(\varepsilon^4)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}N} = -\frac{V_{,\varphi}}{3H^2} + \frac{H}{2\pi}\xi(N)$$

Normalized white noise

$\langle \xi(N) \rangle = 0$

$\langle \xi(N)\xi(N') \rangle = \delta(N - N')$



For slow-roll inflation, curvature power spectrum: $\mathcal{P}_\zeta = \left(\frac{\text{Diffusion}}{\text{Drift}}\right)^2$

Electromagnetic fields during inflation

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_{\rm EM} \right]$$

Motivations for $\mathcal{L}_{\rm EM}$

- Is Inflaton ϕ alone?
- Statistical anisotropy — CMB: $|g_*| \leq 10^{-2}$
- Primordial magnetic field — Blazars: $B_{\rm Mpc} \gtrsim 10^{-16}{\rm G}$
- Galactic magnetic field — Milky Way: $B_{\rm MW} \sim \mu{\rm G}$

Model: Ratra-like coupling $f^2 F^2 +$ Axion-like $F\tilde{F}$ coupling:

$$\mathcal{L}_{\rm EM} \supseteq -\frac{f^2(\phi)}{4} F^{\mu\nu} F_{\mu\nu} - \frac{I^2(\phi)}{4} \gamma F^{\mu\nu} \tilde{F}_{\mu\nu} - \frac{\alpha}{f} \phi F^{\mu\nu} \tilde{F}_{\mu\nu}$$

Stochastic formalism for gauge fields

Unified electric and magnetic field using an auxiliary vector field X_i

Long-Short decomposition:

$$\boldsymbol{X}(t,\boldsymbol{x}) = \boldsymbol{X}^{\mathrm{IR}}(t,\boldsymbol{x}) + \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\Theta\left(k-\varepsilon aH\right)\boldsymbol{X}_{\boldsymbol{k}}(t)\; e^{i\boldsymbol{k}.\boldsymbol{x}}$$

Langevin equation:

$$\boldsymbol{\mathcal{X}}' = b_\nu\ \boldsymbol{\mathcal{X}} + D_\nu(\varepsilon)\ \boldsymbol{\xi}\,, \qquad \boldsymbol{\mathcal{X}} = \frac{\boldsymbol{X}^{IR}}{\sqrt{2\epsilon_H}M_P H}$$

Electromagnetic fields have no classical background values,

$$\boldsymbol{\mathcal{X}}(N) = D_\nu(\varepsilon)\ e^{b_\nu N}\int_0^N e^{-b_\nu s}\mathrm{d}\boldsymbol{W}(s)\,,$$

Ornstein-Uhlenbeck (OU) process: $b_\nu < 0$

frictional drift force $-|b_\nu|\boldsymbol{\mathcal{X}}$ $\sim$ the random force $D_\nu\boldsymbol{\xi}$

Equilibrium state: $\langle\mathcal{X}^2\rangle_{\mathrm{eq}} = \dfrac{3D_\nu^2}{2|b_\nu|}$ at around $N_{\mathrm{eq}} \simeq \mathcal{O}(\ln 10/|b_\nu|)$

Magnetic fields from inflation?

V. Demozzi, V. Mukhanov and H. Rubinstein, JCAP 0908(2009) 025

$$\lambda_B = 1\mathrm{Mpc} \qquad 10^{-9}G \gtrsim B_{\rm obs} \gtrsim 10^{-16}G$$

Weak coupling regime

$$B_{\rm end} \simeq \frac{H^2}{2\pi}\left(\frac{\lambda_{\rm ph}}{H^{-1}}\right)^{n-3}$$

n=2: $B_{\rm now} \simeq 10^{-35}G$

n=2.2: $B_{\rm now} \simeq 10^{-30}G$

n=3: $B_{\rm now} \simeq 10^{-11}G$ ⋆

⋆: Electric back-reaction problem

No back-reaction limit: $n \leq 2.2$

Strong coupling regime

$$B_{\rm end} \simeq \frac{H^2}{2\pi}\left(\frac{\lambda_{\rm ph}}{H^{-1}}\right)^{-n-2}$$

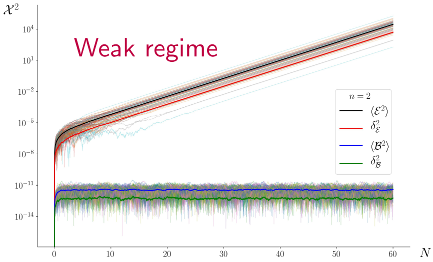
n=-3: $B_{\rm now} \simeq 10^{12}G!$ ⋆

n=-2.2: $B_{\rm now} \simeq 10^{-7}G$

n=-2: $B_{\rm now} \simeq 10^{-11}G$

⋆: Magnetic back-reaction problem

No back-reaction limit: $n \geq -2.2$



$$\boldsymbol{\mathcal{B}}' = -\boldsymbol{\mathcal{B}} + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}}\,\varepsilon\,\boldsymbol{\xi}$$

$$\boldsymbol{\mathcal{E}}' = -\epsilon_H\,\boldsymbol{\mathcal{E}} + \sqrt{6\mathcal{P}_\zeta}\,\boldsymbol{\xi}$$

$$N^B_{\rm eq} \simeq \mathcal{O}(1)$$

$$R \simeq 10^{-10} \ll 1$$

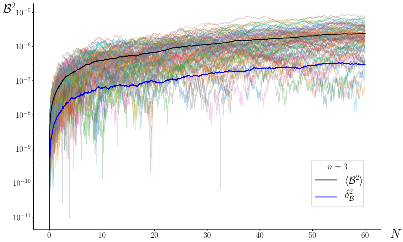
$$B_{\rm now} \simeq 10^{-13}\ {\rm G}$$

Primordial Helical Magnetic Fields from Inflation?

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2111.02147

$$\mathcal{L}_{\rm EM} = -\frac{I^2(\eta)}{4}\left(F^{\mu\nu}F_{\mu\nu} + \gamma F^{\mu\nu}\tilde{F}_{\mu\nu}\right), \qquad I(\eta) \propto a^{-n}$$

The gauge field undergoes tachyonic growth of one of polarizations and leads to generation of a helical magnetic field.



$$\mathcal{B}' = -(2+n)\mathcal{B} + D_B\xi$$
$$\mathcal{E}' = -(2-n)\mathcal{E} - 2\xi\mathcal{B} + D_E\xi$$

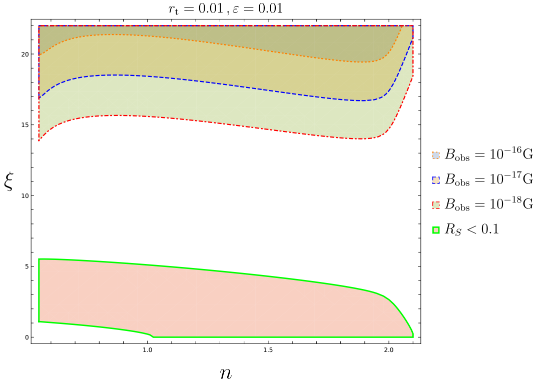
$$\xi = -n\gamma$$

$$D_B \simeq \frac{e^{\pi\xi}\sqrt{\xi}}{\pi\sqrt{3\pi}}\,\frac{\Gamma(2n-1)}{2^n}\,\frac{H}{M_{\rm Pl}}\varepsilon^{3-n}$$

$$D_E \simeq D_B\frac{(2n-1)}{\varepsilon}$$

Primordial Helical Magnetic Fields from Inflation?

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2111.02147



Back-reaction parameter: R_S

Stochastic analysis of Axion inflation and PBH Formation

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

$$\mathcal{L}_{\rm EM} = -\frac{\alpha}{4f}\,\phi F_{\mu\nu}\tilde{F}^{\mu\nu} \;\longrightarrow\; A_\lambda'' + \Big(k^2 - 2\lambda\xi k a H\Big)A_\lambda = 0$$

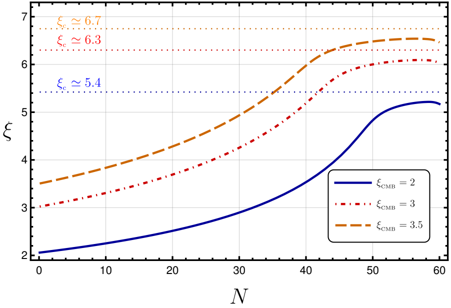
Instability parameter $\xi \equiv \dfrac{\alpha\,\dot\phi}{2fH}$

Assuming $\xi > 0$:

Tachyonic growth of gauge field $A_+(k < 2\xi a H) \propto e^{\pi\xi}$

$$\ddot\phi + 3H\dot\phi - \frac{\nabla^2}{a^2}\phi + V_{,\phi} = J_l$$

Back-reaction on the inflaton field $J_l = \dfrac{\alpha}{f}\boldsymbol{E}\cdot\boldsymbol{B}$



Stochastic analysis of Axion inflation and PBH Formation

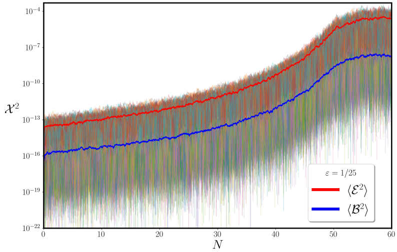
A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

Evolution of the Coarse-grained fields: Long-Short decomposition

$$\phi' = \left(-\frac{V_{,\phi}}{3H^2} + \frac{J_l}{3H^2}\right) + D_\phi \xi \ ,$$

$$\mathcal{E}' = -\left(2 + \left(\frac{\alpha}{f} M_{\rm Pl}\right)^2 \frac{\mathcal{B}^2}{3}\right)\mathcal{E} + \frac{\alpha}{f}\frac{V_{,\phi}}{3H^2}\mathcal{B} + D_E \xi$$

$$\mathcal{B}' = -2\mathcal{B} + D_B \xi$$

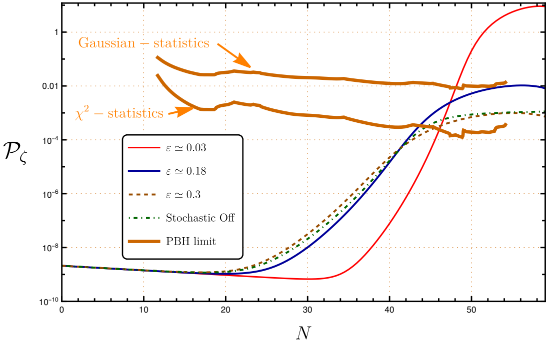


Stochastic analysis of Axion inflation and PBH Formation

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

Scalar mode function:

$$\ddot{\varphi}_{\boldsymbol{k}} + 3H\dot{\varphi}_{\boldsymbol{k}} + \left(\frac{k^2}{a^2} + V_{,\phi\phi}\right)\varphi_{\boldsymbol{k}} = \frac{\alpha}{f}\left(\boldsymbol{E}_l.\boldsymbol{B}_{\boldsymbol{k}} + \boldsymbol{B}_l.\boldsymbol{E}_{\boldsymbol{k}}\right)$$



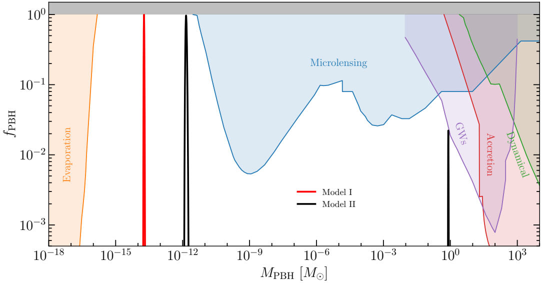
$$\mathcal{P}_\zeta(\xi) \simeq \mathcal{P}_\zeta^{(0)}\left(1 + \frac{\sqrt{2\xi}}{18\pi}\varepsilon^4\mathcal{P}_\zeta^{(0)}\; e^{4\pi\xi}\;\mathcal{G}^2(\varepsilon,\xi)\right)$$

Stochastic analysis of Axion inflation and PBH Formation

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

Enhancement of curvature power spectrum $\longrightarrow$ PBH formation

$$f_{\rm PBH} = \frac{\Omega_{\rm PBH}}{\Omega_{\rm DM}} \simeq 10^8 \; \beta_{\rm PBH} \sqrt{\frac{M_\odot}{M_{\rm PBH}}} \qquad \text{Mass fraction: } \beta_{\rm PBH} = \frac{\rho_{\rm PBH}}{\rho_{\rm tot}}\Big|_{t_{\rm formation}}$$

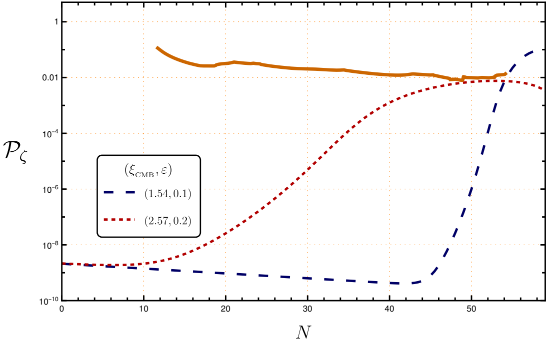


Stochastic analysis of Axion inflation and PBH Formation

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

Formal Approach: Gaussian statistics $\longrightarrow \xi_{\rm CMB} < 1.5$

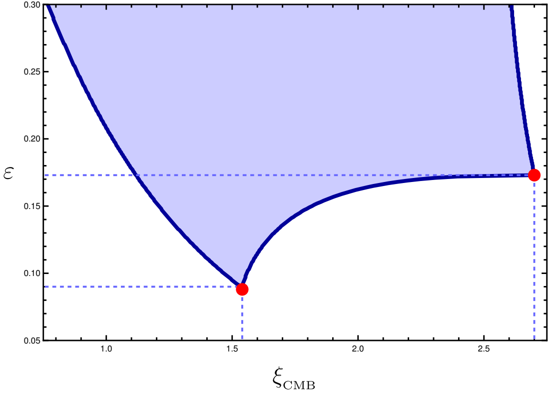
Stochastic Approach: χ^2-statistics $\longrightarrow \xi_{\rm CMB} < 2.5$



Stochastic analysis of Axion inflation and PBH Formation

A. T, A. Nassiri-Rad and H. Firouzjahi, arXiv:2202.02062

Parameter space $(\xi_{\rm CMB}, \epsilon)$



Summary

- The stochastic formalism consists of an effective theory for IR modes of the quantum fields, which are coarse grained at a fixed physical scale.
- The main reason for the amplification of the magnetic field in Ratra model f^2F^2 is due to the Ornstein-Uhlenbeck process which settles the fields into an equilibrium state and prevent them from decaying.
- Tachyonic growth of the magnetic fields in models $F\tilde{F}$ is controlled by a mean-reverting process of stochastic dynamics.
- The stochastic analysis of axion-inflation models shows that the statistic of primordial scalar fluctuation might be changed, e.g., from χ^2 to Gaussian, which is crucial when the models are tested with the PBH constraints.

Future Works

- DM production from stochastic effects during inflation.
- Stochastic effects from massive fields during inflation.
- Stochastic analysis of the non-Gaussian noises.

Thank You For Your Attention!